THE ONLY KÄHLER MANIFOLD WITH DEGREE OF MOBILITY ≥ 3 IS $(\mathbb{C}P(\mathrm{n}), g_{\textbf{Fubini-Study}})$

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Abstract. The degree of mobility of a (pseudo-Riemannian) Kähler metric is the dimension of the space of metrics h-projectively equivalent to it. We prove that a metric on a closed connected manifold can not have the degree of mobility ≥ 3 unless it is essentially the Fubini-Study metric, or the h-projective equivalence is actually the affine equivalence. As the main application we prove an important special case of the classical conjecture attributed to Obata and Yano, stating that a closed manifold admitting an essential group of *h-projective transformations* is $(\mathbb{C}P(n), g_{Fubini-Study})$ (up to multiplication of the metric by a constant). An additional result is the generalization of a certain result of Tanno 1978 for the pseudo-Riemannian situation.

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1. Introduction

1.1. ***h*-planar curves.** Let (g, J) be a Kähler structure on a manifold M^{2n}. We allow the metric g to have arbitrary signature. A curve $\gamma : I \to M^{2n}$ is called *h-planar*, if there exist functions $\alpha(t)$, $\beta(t)$ such that the following ODE holds:

$$\nabla_{\dot\gamma}\dot\gamma = \alpha\dot\gamma + \beta J(\dot\gamma). \tag{1}$$

Actually, equation (1) can be written as an ODE $(\nabla_{\dot\gamma}\dot\gamma) \wedge \dot\gamma \wedge J\dot\gamma = 0$ on γ only; but since this ODE is not in the Euler form, there exist a lot of different h-planar curves with the same initial data $\gamma(t_0), \dot\gamma(t_0)$. Nevertheless, for every chosen functions α and β, equation (1) is an ODE of second order in the Euler form, and has an unique solution with arbitrary initial values $\gamma(t_0), \dot\gamma(t_0)$.

Let us recall basic properties and basic examples of h-planar curves.

Example 1. The property of a curve to be h-planar survives after the reparametrization of the curve. In particular every (reparametrized) geodesic of g is an h-planar curve. This is the reason why $h-$planar curves are also called *almost geodesics* or *complex geodesics* in the literature.

Example 2. Consider a 2-dimensional Riemannian Kähler manifold, i.e. a Riemannian surface (M^2, g) with the induced complex structure J. For this Kähler manifold every curve on M^2 is h-planar, since span$\{\dot\gamma(t), J(\dot\gamma(t))\}$ coincides with the whole $T_{\gamma(t)}M$ for $\dot\gamma(t) \neq 0$.

Example 3. Consider $\mathbb{R}^{2n} = \mathbb{C}^n$ with the standard metric $g = \sum_{j=1}^n dz^j d\bar z^j$ and with the standard complex structure J (acting by multiplication by the imaginary unit i).
Then, a curve γ is h-planar if and only if it lies on a certain "complex line" Span$\{v, J(v)\}$ (for a certain $v \neq \vec{0}$).

Example 4. Consider the complex projective space

$$\mathbb{C}P(n) = \{\text{1-dimensional complex subspaces of } \mathbb{C}^{n+1}\}$$

with the standard complex structure $J = J_{standard}$. The unitary group $U(n+1)$ acts naturally transitively by holomorphic transformations on $\mathbb{C}P(n)$. Since the group $U(n+1)$ is compact, there exists a Kähler metric on $\mathbb{C}P(n)$ invariant with respect to $U(n+1)$. This metric is unique up to multiplication by a constant and is called *the Fubini-Study metric*, we denote it by the symbol g_{FS}. By an appropriate choice of the constant, g_{FS} becomes a Riemannian metric of constant

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holomorphic sectional curvature equal to 1 and we determine g_{FS} uniquely by this choice. Let π be the standard projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P(n)$. We call a subset $L \subseteq \mathbb{C}P(n)$ a *projective line*, if L is the image of a 2-dimensional complex subspace of $\mathbb{C}^{n+1}$ under the projection π.

Let us see that every curve γ lying on a certain projective line L is h-planar (and vice versa). Indeed, L is a totally geodesic 2-dimensional submanifold (for example since there exists an element $f \in U(n+1)$ such that L is the set of fixed points of f). Since L is $J-$invariant, $(L, g_{FS|L}, J_{|L})$ is a two-dimensional Kähler manifold (as in Example 2); in particular every curve on $(L, g_{FS|L}, J_{|L})$ is h-planar. Since the restriction of the connection of g_{FS} to L coincides with the connection of $g_{FS|L}$, every curve h-planar with respect to $(g_{FS|L}, J_{|L})$ is also h-planar with respect to (g_{FS}, J). Now, every initial data $\gamma(0), \dot\gamma(0)$ and every functions $\alpha(t), \beta(t)$ can be realized by a h-planar curve lying on an appropriate projective line. Thus, a curve is h-planar if and only if it lies on a certain projective line L.

1.2. h-projectively equivalent metrics.

Definition 1 (h-projectivity)**.** Two metrics g and $\bar g$ that are Kähler with respect to the same complex structure J are called *h-projectively equivalent*, if each h-planar curve of g is an h-planar curve of $\bar g$ and vice versa.

Example 5. If the metrics g and $\bar g$ are Kähler with respect to the same complex structure J and are *affinely equivalent* (i.e., if their Levi-Civita connections Γ and $\bar\Gamma$ coincide), then they are h-projectively equivalent. Indeed, equation (1) for the first and for the second metric coincides if $\Gamma = \bar\Gamma$.

As we will see further, affine equivalence will be considered as a special *trivial* case of h-projectivity.

Example 6. In particular, for every nondegenerate hermitian matrix $A = (a_{ij}) \in Mat(n, n, \mathbb{C})$ the metric $\bar g = \sum_{i,j=1}^n a_{ij} dz^i d\bar z^j$ is h-projectively equivalent to the metric $g = \sum_{i=1}^n dz^i d\bar z^i$ from Example 3: indeed, the metric $\bar g$ is affinely equivalent to g and is Kähler with respect to the same J. Though there exist other examples of metrics h-projectively equivalent to the metric from Example 3; they can be constructed similar to Example 7.

Let us now construct Kähler metrics h-projectively equivalent to the Fubini-Study metric g_{FS} on $\mathbb{C}P(n)$. The construction is a generalization of the Beltrami's example of projectively equivalent metrics, see [8].

Example 7. Consider a complex linear transformation of $\mathbb{C}^{n+1}$ given by a matrix $A \in GL_{n+1}(\mathbb{C})$ and the induced mapping $f_A : \mathbb{C}P^n \to \mathbb{C}P^n$ defined by $f_A(\pi(x)) = \pi(Ax)$. Since the mapping f_A preserves the complex lines L and since by Example 4 h-planar curves are those lying on a certain projective line L, the pullback $g_A := f_A^* g_{FS}$ is h-projectively equivalent to g_{FS}. For further use let us note that the metric g_A is isometric or affinely equivalent to g_{FS} if and only if A is proportional to a unitary matrix.

1.3. PDE-system for h-projectively equivalent metrics and the degree of mobility.

Let J be a complex structure on M^{2n} and let g and $\bar g$ be two metrics on M^{2n} such that (g, J) and $(\bar g, J)$ are Kähler structures. We consider the following $(0,2)$-tensor a_{ij} on M:

$$a_{ij} = \left(\frac{\det \bar g}{\det g}\right)^{\frac{1}{2(n+1)}} g_{i\alpha} \bar g^{\alpha\beta} g_{\beta j}, \tag{2}$$

where $\bar g^{\alpha\beta}$ is the $(2,0)$-tensor dual to $g_{\alpha\beta}$: $\bar g^{\alpha\beta} \bar g_{\beta\gamma} = \delta^\alpha_\gamma$.
Obviously a_{ij} is a hermitian, symmetric and non-degenerate $(0,2)$-tensor.

Convention. *We work in tensor notations. In particular we denote by "comma" the covariant differentiation with respect to the Levi-Civita connection defined by g, i.e., for example $T_{ij,k} = \nabla_k T_{ij}$ for a (0,2)-tensor T. We sum with respect to repeating indices and use the metric g to raise and lower indices, for example $J_{jk} = g_{j\alpha} J^\alpha{}_k$ is the Kähler 2-form corresponding to g. All indices range from 1 to $2n$; the greek indices $\alpha, \beta, \ldots$ also range from 1 to $2n$ and will be mostly used as*

summation indexes ("dummy" indices in jargon). We also introduce the following notation: for every $1-$*form* ω_i *we denote by* $\bar{\omega}_i = J^{\alpha}{}_{i}\omega_\alpha$ *the "multiplication" of* ω *with the complex structure* J.

The following statement due to Mikes and Domashev plays an important role in the theory of h-projectivity; it reformulates the condition "$\bar{g}$ is h-projectively equivalent to g" to the PDE-language.

Theorem 1 ([40, 41])**.** *Let* (g, J) *and* $(\bar{g}, J)$ *be two Kähler structures on* M^{2n}*. Then,* $\bar{g}$ *is* h*-projectively equivalent to* g *if and only if there exists a* $(0,1)-$*tensor* λ_i *such that* a_{ij} *given by* (2) *satisfies*

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - \bar{\lambda}_i J_{jk} - \bar{\lambda}_j J_{ik} \tag{3}$$

One can and should regard equation (3) as a PDE-system on the unknown (a_{ij}, λ_i) whose coefficients depend on the metric g. Let us mention though that it is possible to consider (3) as a PDE-system on the unknown (a_{ij}) only: Indeed, contracting (3) with g^{ij} we obtain $\left(a^i_i\right)_{,k} = 4\lambda_k$ (which in particular implies that the covector λ_i is a gradient, i.e., $\lambda_{i,j} = \lambda_{j,i}$).

Note that the formula (2) is invertible. Then, the set of the metrics $\bar{g}$ h-projectively equivalent to g is essentially the same as the set of the hermitian and symmetric solutions of (3) (the only difference is the case when a_{ij} is degenerate; but since adding const $\cdot\, g_{ij}$ to a_{ij} does not change the property of a_{ij} to be a solution, this difference is not important). Indeed, one can show that if (g, J) is Kähler, a_{ij} is hermitian, symmetric, nondegenerate and satisfies (3) for a certain λ_i, then the metric $\bar{g}$ constructed via (2) is also Kähler with respect to J.

We see that the PDE-system (3) is linear, hence the set of its solutions is a linear vector space.

Definition 2. The *degree of mobility* of a Kähler metric g is the dimension of the space of solutions (a_{ij}, λ_i) of (3), where a_{ij} is symmetric and hermitian.

Remark 1. The degree of mobility D is at least 1 and is finite (assuming $\dim(M) \geq 4$; in the two-dimensional case, every two conformally equivalent metrics are h-projectively equivalent), $1 \leq D < \infty$. Indeed, g itself is always a solution of (3) (with $\lambda_i \equiv 0$), implying $D \geq 1$. We will not make use of the fact that D is finite, in fact $D \leq (n+1)^2$, but it will be a direct consequence of Section 4 (and follows for example from [41, Theorem 2]).

Convention. *The equation (3) plays a fundamental role in our paper. Whenever we speak about a solution* (a_{ij}, λ_i) *of this equation, we assume that* a_{ij} *is symmetric and hermitian. One of the reasons for it is that if* a_{ij} *is constructed by* (2)*, then it is automatically symmetric and hermitian. The second reason is that the procedure of symmetrization and hermitization*

$$T_{ij} \mapsto \frac{1}{4} T_{\alpha\beta}\left(\delta^\alpha_i \delta^\beta_j + \delta^\alpha_j \delta^\beta_i + J^{\alpha}{}_{i} J^{\beta}{}_{j} + J^{\alpha}{}_{j} J^{\beta}{}_{i}\right)$$

does not affect the right-hand side of the equation; so without loss of generality we can always think that a_{ij} *in (3) is symmetric and hermitian.*

Remark 2. For further use, let us note that if $\lambda_i \equiv 0$, then the metric $\bar{g}$ corresponding to a_{ij} is affinely equivalent to g (if it exists, i.e., if a_{ij} is nondegenerate).

1.4. **Main result.** Our main result is the following

Theorem 2. *Let* (M^{2n}, g, J) *be a closed connected Kähler manifold of degree of mobility* $D \geq 3$ *and of real dimension* $2n \geq 4$*. Then*

- *there is a constant* $c \in \mathbb{R}$*,* $c \neq 0$*, such that* $(M^{2n}, c \cdot g, J)$ *is* $(\mathbb{C}P(n), g_{FS}, J_{standard})$ *where* g_{FS} *denotes the Fubini-Study metric on* $\mathbb{C}P(n)$ *with the standard complex structure*

  or

- *each Kähler metric* $\bar{g}$*,* h*-projectively equivalent to* g*, is affine equivalent to* g*.*

In other words, a closed Kähler manifold (M^{2n}, g, J) which is not (a quotient of) $(\mathbb{C}P(n), \mathrm{const} \cdot g_{FS}, J_{standard})$ can not have $D \geq 3$ unless every metric h-projectively equivalent to g is affinely equivalent to g.

We would like to point out that we do not assume in Theorem 2 that the metric g is Riemannian: an essential part of the proof is to show that it must be definite (i.e., that $\mathrm{const} \cdot g$ is Riemannian for an appropriate constant).

1.5. **All conditions in Theorem 2 are necessary.** The assumption $D \geq 3$ is necessary. Indeed, a construction of a Kähler metric g on $\mathbb{C}P(n)$ of non-constant holomorphic sectional curvature such that it admits a metric $\bar{g}$ that is h-projectively equivalent to g, but not affinely equivalent to g can be extracted from [20]. In a certain sense, Kiyohara found a way how one can perturb a pair of h-projectively equivalent metrics on a closed manifold such that they remain h-projectively equivalent. The space of perturbations is big and depends on functional parameters. Perturbing h-projectively equivalent metrics from Example 7, we obtain (for generic parameters of the perturbation) metrics on $\mathbb{C}P(n)$ of non-constant holomorphic sectional curvature admitting non-trivial h-projectivity. More examples can be extracted from [5], see discussion at the end of Section 1.6.1.

The assumption that the manifold is closed is also necessary. The simplest examples of local metrics different from g_{FS} with big degree of mobility are due to [54], see also [11, 50]: it was shown that (locally) a metric of constant holomorphic curvature (even if the metric is not positive definite and the sign of the curvature is negative) admits a huge space of h-projectively equivalent metrics. One can also construct examples of (local) metrics of non-constant holomorphic curvature with degree of mobility ≥ 3 using the results of [39, §2.2].

The second possibility in Theorem 2 (when g and $\bar{g}$ are affinely equivalent) is also necessary. Indeed, consider the direct product of three Kähler manifolds

$$(M_1, g_1, J_1) \times (M_2, g_2, J_2) \times (M_3, g_3, J_3).$$

It is a Kähler manifold diffeomorphic to the product $M_1 \times M_2 \times M_3$, the metric is the sum of the metrics $g_1 + g_2 + g_3$, and the complex structure is the sum of the complex structures. Then, for any constants $c_1, c_2, c_3 \neq 0$, the metrics $c_1 \cdot g_1 + c_2 \cdot g_2 + c_2 \cdot g_3$ is h-projectively equivalent to $g_1 + g_2 + g_3$ (because they are affinely equivalent to it), i.e., the degree of mobility of $g_1 + g_2 + g_3$ is at least 3. If M_i are closed, then $M_1 \times M_2 \times M_3$ is closed as well. Of course, the metric $g_1 + g_2 + g_3$ is not $\mathrm{const} \cdot g_{FS}$.

1.6. **History, motivation, and first applications.**

1.6.1. *History and motivation.* h-planar curves and h-projectivity of Kähler metrics where introduced in [44, §§9-10]. Otsuki and Tashiro did not explain explicitly their motivation, from the context one may suppose that they tried to study projectively equivalent metrics (the definition is in Section 1.10) in the Kähler situation, found out that they are not interesting (impossible except of few trivial examples), and suggested a Kähler analog of projectively equivalent metrics. Actually, it was one of the main trend of their time to adapt Riemannian objects to the Kähler situation, see for example the book [59] (where many objects were generalized to the Kähler situation; $h-$projectively equivalent metrics are in the last chapter of this book).

The notion turned out to be interesting and successful, there are a lot of papers studying h-projectivity and its generalizations, see for example the recent survey [39]. At a certain period of time $h-$projectivity was one of the main research topics of the Japanese and Soviet (mostly Odessa and Kazan) geometry schools. At least two books, [48] and [59], have chapters on $h-$projectively equivalent metrics.

One of the mainstreams in the theory of h-projectivity is to understand the group of *h-projective transformations*, i.e., the group of diffeomorphisms of (M^{2n}, g, J) that preserve the complex structure and send the metric to a metric that is h-projectively equivalent to g. This set is obviously a group, Ishihara [18] and Yoshimatsu [61] have shown that it is a finite dimensional Lie group and the challenge was to understand the codimension of the group of affine transformations or isometries in this group, see for example [18, 16, 60, 1, 14, 39].

As it follows from Example 7, the group of h-projective transformations of $(\mathbb{C}P(n), g_{FS}, J_{standard})$ is much bigger than its subgroup of affine transformations. A classical conjecture (in folklore this conjecture is attributed to Obata and Yano, though we did not find a reference where they formulate it explicitly) says that, *on closed Riemannian Kähler manifolds that are not* $(\mathbb{C}P(n), \mathrm{const} \cdot g_{FS}, J_{standard})$*, the connected component of the group of h-projective transformations contains isometries only.* In particular, in the above mentioned papers [18, 17, 16, 60, 1], the conjecture was proved under certain additional assumptions; for example, the additional assumption in [16, 60, 1] was that the scalar curvature of the metric is constant.

For the Riemannian metrics, the Yano-Obata conjecture was proved in the recent paper [37]. The proof uses different techniques to those employed in the present article, but does rely in part on certain results (Theorem 2 and Section 2.5) of the present paper.

In Section 1.6.2, we give new results assuming that the metric has arbitrary signature. In particular, we show that the codimension of the subgroup of isometries in the group of h-projective transformation is at most one.

Recent interest to h-projectivity is in particular due to an unexpected connection between h-projectively equivalent metrics and integrable geodesic flows: it appears that the existence of $\bar{g}$ h-projectively equivalent to g allows to construct quadratic and linear integrals for the geodesic flow of g, see for example [56, 21]. Theorem 2 shows that there is no metric (except of Fubini-Study) on a closed Kähler manifold such that its geodesic flow is superintegrable with integrals coming from h-projectively equivalent metrics.

Additional interests to h-projective equivalence is due to its connection with the so called *hamiltonian 2-forms* defined and investigated in Apostolov et al [4, 5, 6, 7]. It is easy to see that a hamiltonian 2-form is essentially the same as a h-projectively equivalent metric $\bar{g}$, since the defining equation [4, equation (12)] of a hamiltonian 2-form is algebraically equivalent to the equation (3) from Theorem 1. The motivation of Apostolov et al to study hamiltonian 2-forms is different from that of Otsuki and Tashiro and is explained in [4, 5]. Roughly speaking, they observed that many interesting problems on Kähler manifolds lead to hamiltonian 2-forms and suggested to study them. The motivation is justified in [6, 7], where they indeed constructed new interesting and useful examples of Kähler manifolds. There is also a direct connection between h-projectively equivalent metrics and conformal Killing (or twistor) 2-forms studied in [42, 46, 47], see Appendix A of [4] for details.

In private communications with the authors of [4, 5, 6, 7] we got informed that they did not know that the object they considered was studied before under another name. Indeed, they re-derived certain facts that were well known in the theory of h-projectively equivalent metrics. On the other hand, the papers [4, 5, 6, 7] contain several solutions of the problems studied in the framework of h-projectively equivalent metrics; in particular they gave a global description of metrics admitting hamiltonian 2-forms providing us with new nontrivial examples of h-projectively equivalent metrics.

1.6.2. *First applications: special case of the Yano-Obata conjecture.* Let (M^{2n}, g, J) be a Kähler manifold. Recall that a diffeomorphism $f : M \to M$ is called a *h-projective transformation* if it preserves the complex structure J and sends the metric g to a metric that is h-projectively equivalent to g. The set of all h-projective transformations of (M^{2n}, g, J) forms a Lie group which we denote by HProj. We denote by HProj_0 its connected component containing the identity. The groups of affine transformations and isometries of M preserving the complex structure and their connected components containing the identity will be denoted by $\mathrm{Aff}(g, J)$, $\mathrm{Iso}(g, J)$, $\mathrm{Aff}_0(g, J)$, and $\mathrm{Iso}_0(g, J)$, respectively.

Corollary 1. *Let (M^{2n}, g, J) be a closed connected Kähler manifold of dimension $2n \geq 4$. Assume that for every* $\mathrm{const} \neq 0$ *the manifold (M^{2n}, g, J) is not $(\mathbb{C}P(n), \mathrm{const} \cdot g_{FS}, J_{standard})$. Then the group $Iso_0(g, J)$ has the codimension at most one in the group $HProj_0$, or $HProj = Aff(g, J)$.*

Proof. First assume $D = 1$. This means that each metric that is h-projectively equivalent to g, is proportional to it. Thus, every h-projective transformation is a homothety. Since the manifold is closed, every homothety is an isometry implying $\mathrm{HProj} = \mathrm{Iso}(g, J)$.

Assume now $D \geq 3$. Then, by Theorem 2, every $\bar{g}$ h-projectively equivalent to g is affinely equivalent to g implying $\mathrm{HProj} = \mathrm{Aff}(g, J)$.

The remaining case is $D = 2$. We need to show that the Lie-algebra of $\mathrm{Iso}_0(g, J)$ has codimension at most one in the Lie-algebra of HProj_0.

Let u, v be infinitesimal h-projective transformations, i.e. vector fields on M generating 1-parameter groups of h-projective transformations. We need to show that their certain linear combination is a Killing vector field. Let us first construct a mapping $\Psi : u \mapsto a_u$ sending an infinitesimal h-projective transformation to a solution of (3).

We denote by Φ_t^u the flow of u and define $g_t := (\Phi_t^u)^* g$. As we recalled in Section 1.3 (see Theorem 1 there), the $(0,2)-$tensor $a(t)_{ij}$ given by (in matrix notation)

$$a(t) = \left(\frac{\det g_t}{\det g}\right)^{\frac{1}{2(n+1)}} g g_t^{-1} g$$

satisfies equation (3). Taking the derivative at $t = 0$, and replacing the $t-$derivatives of tensors by Lie derivatives, we obtain that the $(0,2)-$tensor

$$a_u := L_u g - \frac{\operatorname{trace} g^{-1} L_u g}{2(n+1)} g$$

satisfies equation (3).

We define then the mapping Ψ by $\Psi(u) = a_u$. The mapping is clearly linear in u. Since the two-dimensional space of the solutions of (3) contains the one-dimensional subspace $\{c \cdot g \mid c \in \mathbb{R}\}$, for every two infinitesimal h-projective transformations u, v there exists a linear combination $bu + dv$ such that $\Psi(bu + dv) = cg$ (for a certain $c \in \mathbb{R}$). Let us show that $bu + dv$ is a Killing vector field. We have:

$$L_{bu+dv} g - \frac{\operatorname{trace} g^{-1} L_{bu+dv} g}{2(n+1)} g = cg. \tag{4}$$

Multiplying this (matrix) equation by the inverse matrix of g and taking the trace, we obtain

$$\operatorname{trace}(g^{-1} L_{bu+dv} g) - \tfrac{2n}{2(n+1)} \operatorname{trace}(g^{-1} L_{bu+dv} g) = 2nc.$$

Thus, $\operatorname{trace}(g^{-1} L_{bu+dv} g) = 2n(n+1)c$. Substituting this in (4), we obtain that $L_{bu+dv} g = c(1-n) \cdot g$. Then, $bu + dv$ is an infinitesimal homothety. Since the manifold is closed, any infinitesimal homothety is a Killing vector field implying that $bu + dv$ is a Killing vector field as we claimed. □

1.7. **Additional motivation: new methods for the investigation of the global behavior of h-projectively equivalent pseudo-Riemannian metrics.** In many cases, local statements about Riemannian metrics could be generalised for the pseudo-Riemannian setting, though sometimes this generalisation is difficult. As a rule, it is very difficult to generalize global statements about Riemannian metrics to the pseudo-Riemannian setting. The theory of h-projectively equivalent metrics is not an exception: certain local results could be generalized without essential difficulties. Up to now, no global (say if the manifold is closed) methods for the investigation of h-projectively equivalent metrics were generalized for the pseudo-Riemannian setting.

More precisely, virtually every global result (see for example the surveys [39, 49]) on $h-$projectively equivalent Riemannian metrics was obtained by using the so-called "Bochner technique", which requires that the metric is positively defined.

Our proofs (we explain the scheme in Section 1.9) use essentially new methods (in Section 1.10 we explain that these methods were motivated by new results in the theory of projectively equivalent metrics). We expect further applications of these new methods in the theory of h-projectively equivalent metrics, and in other parts of differential geometry.

1.8. **Additional result: Tanno-Theorem for pseudo-Riemannian metrics.** Let us recall the following classical result of Tanno and Hiramatu:

Theorem 3 ([53],[16])**.** *Let f be a non-constant smooth function on a closed Riemannian Kähler manifold (M^{2n}, g, J) of dimension $2n \geq 4$ such that the equation*

$$f_{,ijk} = \kappa(2f_{,k} \cdot g_{ij} + f_{,i} \cdot g_{jk} + f_{,j} \cdot g_{ik} - \bar{f}_{,i} \cdot J_{jk} - \bar{f}_{,j} \cdot J_{ik}). \tag{5}$$

is fulfilled (for a certain constant κ). Then, $\kappa < 0$ and (M^{2n}, g, J) has constant holomorphic sectional curvature -4κ. In particular, $(M^{2n}, -4\kappa \cdot g, J)$ is $(\mathbb{C}P(n), g_{FS}, J_{standard})$.

More precisely, Tanno [53] assumed that $\kappa < 0$; in this case it is sufficient to require that the manifold is complete. Hiramatu [16] proved that the equation can not have nonconstant solutions for $\kappa \geq 0$, if the manifold is closed. One can construct counterexamples to the latter statement, if the manifold is merely complete.

We will show in Section 6 that a part of the proof of our main result gives also a proof of the pseudo-Riemannian version of the above statement:

Theorem 4. *Let f be a non-constant smooth function on a closed connected pseudo-Riemannian Kähler manifold (M^{2n}, g, J) of dimension $2n \geq 4$ such that the equation* (5) *is fulfilled (for a certain constant κ). Then, $\kappa \neq 0$ and $(M^{2n}, -4\kappa \cdot g, J)$ is $(\mathbb{C}P(n), g_{FS}, J_{standard})$.*

1.9. **Plan of the proof.** We assume that (M^{2n}, g, J) is a closed connected Kähler manifold of dimension $2n \geq 4$. We divide the proof of Theorem 2 in four steps.

- In Section 2, assuming $D \geq 3$, we show that for every solution (a_{ij}, λ_i) of equation (3) there exists a constant $B \in \mathbb{R}$ and a function μ such that the following "extended" system

$$\begin{aligned} a_{ij,k} &= \lambda_i g_{jk} + \lambda_j g_{ik} - \bar{\lambda}_i J_{jk} - \bar{\lambda}_j J_{ik} \\ \lambda_{i,j} &= \mu g_{ij} + B a_{ij} \\ \mu_{,i} &= 2B\lambda_i \end{aligned}$$

  is satisfied (see Theorem 5). For $a_{ij} \neq \mathrm{const} \cdot g_{ij}$, the constant B is uniquely determined by the metric (Corollary 5), i.e., is the same for all solutions of (3) that are not proportional to g.

  In Sections 3, 4, 5 we will work with the above "extended" system only, i.e., we will not use that the degree of mobility of g is ≥ 3 anymore. We show that *the existence of a solution (a_{ij}, λ_i, μ) with $\lambda_i \not\equiv 0$ on a closed connected Kähler manifold implies that the metric is proportional to the Fubini-Study metric.* We proceed as follows:
- In Section 3 (see Theorem 6), we show that $B \neq 0$ unless $\lambda_i \equiv 0$.
- If $B \neq 0$, by replacing g with $-B \cdot g$, without loss of generality we can assume $B = -1$. In Section 4, we show that, for $B = -1$, the metric g is positively definite.
- In Section 5, we combine the results of the previous sections and the result of Tanno [53] we recalled in Section 1.8, to show that our manifold is $(\mathbb{C}P(n), \mathrm{const} \cdot g_{FS}, J_{standard})$. This concludes the proof of Theorem 2.

1.10. **Relation with projective equivalence.** Two metrics g and $\bar{g}$ on the same manifold are *projectively equivalent*, if every geodesic of g, after an appropriate reparametrization, is a geodesic of $\bar{g}$. As we already mentioned in Section 1.6.1, we think that the notion "h-projective equivalence" was introduced as an attempt to adapt the notion "projective equivalence" to Kähler metrics. It is therefore not a surprise that certain methods from the theory of projectively equivalent metrics could be adapted for the h-projective questions. For example, the above mentioned papers [16, 60, 1] are actually an h-projective analog of the papers [57, 15] (dealing with projective transformations), see also [13, 51]. Moreover, [61, 54] are "Kählerizations" of [17, 52], and many results listed in the survey [39] are "Kählerizations" of those listed in [38].

The Yano-Obata conjecture is also an h-projective analog of the so-called projective Lichnerowicz-Obata conjecture (recently proved in [31, 28], see also [26, 27]). There also exists a conformal analog of this conjecture (the so called conformal Lichnerowicz-Obata conjecture proved in [2, 43, 45]), whose $CR-$analog was proved in [45], and finsler analog in [35].

We also used certain ideas from the theory of projectively equivalent metrics. In particular, the scheme of the first part of the proof of Theorem 2 is close to the scheme of the proof of [19,

Theorem 1], see also [29], the scheme of the second part of the proof is close to the proof of [33, Theorem 1] (though the proofs in the present paper are technically much more complicated than the proofs in [19, 33]).

Let us also recall that recently new methods for the investigation of projectively equivalent metrics were suggested. A group of these new methods came from the theory of integrable systems and from the dynamical systems [23, 34, 24, 25]. We expect that these methods could also be adapted for the investigation of h-projectively equivalent metrics (first steps were already done in [21]). Another group of new methods came from the geometric theory of ODEs, see for example [10, 30, 9]. We expect that these methods could also be adapted for h-projective transformations.

Let us also recall that equation (5) was introduced in [53] as "Kählerization" of $f_{,ijk} = \kappa(2f_{,k} \cdot g_{ij} + f_{,i} \cdot g_{jk} + f_{,j} \cdot g_{ik})$. The latter equation appeared independently and was helpful in many parts of differential geometry: in spectral geometry [53, 12], in cone geometry [12, 3], and in conformal and projective geometry (see [15, 53] and [32, 33] for references). We expect that equation (5) will be helpful in the "Kählerizations" of these geometries.

2. Local theory and extended system

The goal of Section 2 is to prove the following

Theorem 5. *Let (M^{2n}, J, g) be a connected Kähler manifold of dimension $2n \geq 4$. If the degree of mobility D of g is ≥ 3, then for every solution (a_{ij}, λ_i) of (3), such that $a_{ij} \neq const \cdot g$, there exists a unique constant B and a scalar function μ, such that the extended system*

$$\begin{array}{ll} a_{ij,k} & = \lambda_i g_{jk} + \lambda_j g_{ik} - \bar{\lambda}_i J_{jk} - \bar{\lambda}_j J_{ik} \\ \lambda_{i,j} & = \mu g_{ij} + B a_{ij} \\ \mu_{,i} & = 2B\lambda_i \end{array} \tag{6}$$

is satisfied.

We see that the first equation of (6) is precisely the equation (3), i.e., is fulfilled by assumptions. We would like to note here that the second and the third equations are *not* differential consequences of the first one: they require the assumption that the degree of mobility is ≥ 3.

The proof of the second equation is the lengthiest and trickiest part of the proof of Theorem 5. After recalling basic properties of λ_i in Section 2.1, we will first prove a pure algebraic result (Lemma 2). Together with Lemma 5, it will imply that the equation $\lambda_{i,j} = \mu g_{ij} + B a_{ij}$ holds in a neighborhood of almost every point of M for a certain *function* B. Then, in Lemma 6 we show that, locally, in a neighborhood of almost every point, the function B is actually a constant. The constant B and the function μ could a priori depend on a neighborhood of the manifold, the last step will be to show that B and μ are the same for each neighborhood and, hence, are globally defined (Section 2.5). Now, the third equation of Theorem 5 will be obtained as a differential corollary of the first two.

Note also that $(g_{ij}, 0)$ is also a solution of (6), with $\mu = -B$, so Theorem 5 holds for this solution except for the constant B is not unique anymore. In Section 4 we will consider $(g_{ij}, 0)$ as a solution of (6) with $B = -1$ and $\mu = 1$.

2.1. Killing vector field for the geodesic flow of g.

In this section we show that the 1-form $\bar{\lambda}_i$ satisfies the Killing equation, a fact which we shall use several times during our paper.

Lemma 1 (Folklore). *Let (M^{2n}, g, J) be a Kähler manifold of dimension $2n \geq 4$ and let (a_{ij}, λ_i) be a solution of equation (3). Then J anticommutes with g_{ij}, a_{ij} and $\lambda_{i,j}$:*

$$J^\alpha{}_i g_{\alpha j} = -g_{i\alpha} J^\alpha{}_j,$$
$$J^\alpha{}_i a_{\alpha j} = -a_{i\alpha} J^\alpha{}_j,$$
$$J^\alpha{}_i \lambda_{\alpha,j} = -\lambda_{i,\alpha} J^\alpha{}_j.$$

Proof. The first equality is a part of the definition of Kähler metrics, the second property follows from our convention from Section 1.3. The third equality is also somehow known: it follows

immediately from [41, equation (13)] and [4, Proposition 3]. For the convenience of the reader, we give its proof but it does not pretend to be new.

Differentiating (3), we obtain

$$a_{ij,kl} = \lambda_{i,l}g_{jk} + \lambda_{j,l}g_{ik} - \bar\lambda_{i,l}J_{jk} - \bar\lambda_{j,l}J_{ik}.$$

Substituting this into the formula $a_{ij,kl} - a_{ij,lk} = R^r_{ikl}a_{rj} + R^r_{jkl}a_{ir}$ (which is fulfilled for every $(0,2)$-tensor a_{ij}) we obtain

$$\begin{aligned} a_{ij,kl} - a_{ij,lk} &= \lambda_{i,l}g_{jk} - \lambda_{i,k}g_{jl} + \lambda_{j,l}g_{ik} - \lambda_{j,k}g_{il} - \bar\lambda_{i,l}J_{jk} + \bar\lambda_{i,k}J_{jl} - \bar\lambda_{j,l}J_{ik} + \bar\lambda_{j,k}J_{il} \\ &= R^r_{ikl}a_{rj} + R^r_{jkl}a_{ir}. \end{aligned} \tag{7}$$

Multiplying this equation with g^{jk} and summing with respect to repeating indices, we obtain:

$$\begin{aligned} &2n\lambda_{i,l} - \lambda_{i,l} + \lambda_{i,l} - g^{jk}\lambda_{j,k}g_{il} - 0 + \bar\lambda_{i,k}J^k{}_l - \lambda_{i,l} + g^{jk}\bar\lambda_{j,k}J_{il} \\ &= (2n-1)\lambda_{i,l} - g^{jk}\lambda_{j,k}g_{il} + \bar\lambda_{i,k}J^k{}_l + g^{jk}\bar\lambda_{j,k}J_{il} = g^{jk}R^r_{ikl}a_{rj} + g^{jk}R^r_{jkl}a_{ir}. \end{aligned} \tag{8}$$

Recall that g_{ij} and a_{ij} are hermitian and the curvature satisfies the symmetry relations

$$R^i_{\alpha kl}J^\alpha{}_j = J^i{}_\alpha R^\alpha_{jkl} \text{ and } R^i_{j\alpha\beta}J^\alpha{}_kJ^\beta{}_l = R^i_{jkl}.$$

Now, let us rename $i \to i'$ and $l \to l'$, multiply equation (8) by $J^{i'}{}_iJ^{l'}{}_l$, and sum with respect to repeating indices. We want to show that this operation does not change the right-hand side of the equation. First we consider the second term on the right-hand side:

$$\begin{aligned} g^{jk}R^r_{jkl'}a_{i'r}J^{i'}{}_iJ^{l'}{}_l = -g^{jk}R^r_{jkl'}a_{i'i}J^{i'}{}_rJ^{l'}{}_l = -g^{jk}R^{i'}_{rkl'}a_{i'i}J^r{}_jJ^{l'}{}_l = -g^{jk}R^r_{i'kl'}a_{ri}J^{i'}{}_jJ^{l'}{}_l \\ = g^{ji'}R^r_{i'kl'}a_{ri}J^k{}_jJ^{l'}{}_l = g^{ji'}R^r_{i'jl}a_{ri} = g^{jk}R^r_{jkl}a_{ir} \end{aligned}$$

We see that this term remains unchanged. Similarly, for the first term on the right-hand side we have

$$\begin{aligned} g^{jk}R^r_{i'kl'}a_{rj}J^{i'}{}_iJ^{l'}{}_l = g^{jk}R^{i'}_{ikl'}a_{rj}J^r{}_{i'}J^{l'}{}_l = -g^{jk}R^{i'}_{ikl'}a_{ri'}J^r{}_jJ^{l'}{}_l \\ = g^{jr}R^{i'}_{ikl'}a_{ri'}J^k{}_jJ^{l'}{}_l = g^{jr}R^{i'}_{ijl}a_{ri'} = g^{jk}R^r_{ikl}a_{jr}, \end{aligned}$$

which again shows that the operation above does not change this term. Thus, the right-hand side of (8) remains unchanged, so the difference of the left-hand side of (8) and the transformed left-hand side of (8) must be zero. We obtain:

$$\begin{aligned} 0 = (2n-1)\lambda_{i',l'}J^{i'}{}_iJ^{l'}{}_l - g^{jk}\lambda_{j,k}g_{i'l'}J^{i'}{}_iJ^{l'}{}_l + \bar\lambda_{i',k}J^k{}_{l'}J^{i'}{}_iJ^{l'}{}_l + g^{jk}\bar\lambda_{j,k}J_{i'l'}J^{i'}{}_iJ^{l'}{}_l \\ -(2n-1)\lambda_{i,l} + g^{jk}\lambda_{j,k}g_{il} - \bar\lambda_{i,k}J^k{}_l - g^{jk}\bar\lambda_{j,k}J_{il} \\ = (2n-1)\bar\lambda_{i,k}J^k{}_l - (2n-1)\lambda_{i,l} - \bar\lambda_{i,k}J^k{}_l - \bar\lambda_{i',l}J^{i'}{}_i = (2n-2)(\bar\lambda_{i,k}J^k{}_l - \lambda_{i,l}) \end{aligned}$$

Hence, $\bar\lambda_{i,k}J^k{}_l = \lambda_{i,l}$. Multiplying by $J^l{}_j$ and using that $\lambda_{i,j}$ is symmetric yields the desired formula $-\bar\lambda_{i,j} = \lambda_{i,l}J^l{}_j = \lambda_{l,i}J^l{}_j = \bar\lambda_{j,i}$. □

Corollary 2 ([4])**.** *Let (M^{2n}, g, J) be a Kähler manifold of dimension $2n \geq 4$. If (a_{ij}, λ_i) is a solution of equation (3), then $\bar\lambda^i := g^{i\alpha}\bar\lambda_\alpha$ is a Killing vector field for g.*

Proof. A vector field v^i is Killing, if and only if the Killing equation $v_{i,j} + v_{j,i} = 0$ is satisfied. For the vector field $\bar\lambda^i$, the Killing equation reads $\bar\lambda_{i,j} + \bar\lambda_{j,i} = 0$ and is equivalent to the third equality of Lemma 1. □

Corollary 3. *Let (a_{ij}, λ_i) be a solution of equation (3) on a connected Kähler manifold (M^{2n}, g, J) of dimension $2n \geq 4$. If $\lambda_i \neq 0$ at a point, then $\lambda_i \neq 0$ at almost every point.*

Convention. *Within the whole paper we understand "almost everywhere" and "almost every" in the topological sense: a condition is fulfilled almost everywhere (or in almost every point) if and only if the set of the points where it is fulfilled is dense in M.*

Proof. If $\lambda_i \neq 0$ at a point, then the Killing vector field $\bar\lambda^i$ is not identically zero. It is known that a Killing vector field that is not identically zero does not vanish on an open nonempty subset (to see it one can use the fact that the flow of a Killing vector field commutes with the exponential mapping). Thus, $\bar\lambda^i \neq 0$ at almost every point, implying $\lambda_i \neq 0$ at almost every point. □

Corollary 4. *Let (M^{2n}, g, J) be a connected Kähler manifold of dimension $2n \geq 4$ and let (a_{ij}, λ_i) be a solution of (3) such that $a_{ij} = 0$ at every point of some open subset $U \subseteq M$. Then $(a_{ij}, \lambda_i) \equiv (0, 0)$ on the whole M.*

Proof. If $a_{ij} \equiv 0$ in U, then $\lambda_i \equiv 0$ in U implying $\lambda_i \equiv 0$ on the whole M in view of Corollary 3. Then, equation (3) implies that a_{ij} is covariantly constant on M. Since it vanishes at a point, it vanishes everywhere. □

2.2. **Algebraic lemma.** Let us denote by $\mathcal{J}$ the following $(2,2)$-tensor:

$$\mathcal{J}_{ij}^{\alpha\beta} = \delta_i^\alpha \delta_j^\beta + J^\alpha{}_i J^\beta{}_j. \tag{9}$$

Using this notation one can rewrite equation (3) in the form

$$a_{ij,k} = \mathcal{J}_{ij}^{i'j'}(\lambda_{i'} g_{j'k} + \lambda_{j'} g_{i'k}) \tag{10}$$

The first step in the proof of Theorem 5 will be to show the validity of the second equation of the system (6) in a point:

Lemma 2. *Let (M^{2n}, g, J) be a Kähler manifold of dimension $2n \geq 4$ and let (a_{ij}, λ_i) and (A_{ij}, Λ_i) be solutions of (3) such that at the point $p \in M$, a, g, and A are linearly independent. Then, there exist numbers B and μ, such that the equation*

$$\lambda_{i,j} = \mu g_{ij} + B a_{ij}. \tag{11}$$

holds at p.

Proof. Substituting (10) in $a_{ij,kl} - a_{ij,lk} = a_{i\alpha} R^\alpha_{jkl} + a_{j\alpha} R^\alpha_{ikl}$, we obtain

$$a_{i\alpha} R^\alpha_{jkl} + a_{j\alpha} R^\alpha_{ikl} = \mathcal{J}_{ij}^{i'j'}(\lambda_{l,i'} g_{j'k} + \lambda_{l,j'} g_{i'k} - \lambda_{k,i'} g_{j'l} - \lambda_{k,j'} g_{i'l}) \tag{12}$$

These equations are fulfilled for every solution of (3), thus for (A_{ij}, Λ_i). We denote by (12.A) the equation (12) with (a_{ij}, λ_i) replaced by (A_{ij}, Λ_i). From this point we will work in the tangent space to the fixed point p only.

Since equations (12) and (12.A) are not affected by the transformation (for any constants a, A, c, C)

$$a_{ij} \to a_{ij} + a \cdot g_{ij}, \qquad \lambda_{i,j} \to \lambda_{i,j} + c \cdot g_{ij}, \tag{13}$$

$$A_{ij} \to A_{ij} + A \cdot g_{ij}, \qquad \Lambda_{i,j} \to \Lambda_{i,j} + C \cdot g_{ij}, \tag{14}$$

without loss of generality we can assume that a_{ij}, $\lambda_{i,j}$, A_{ij} and $\Lambda_{i,j}$ are trace-free, i.e.

$$a_{ij} g^{ij} = \lambda_{i,j} g^{ij} = A_{ij} g^{ij} = \Lambda_{i,j} g^{ij} = 0. \tag{15}$$

In this “trace-free” situation, our goal is to show that $\lambda_{i,j} = B \cdot a_{ij}$ for a certain number B.

After contracting (12) with $A^l_{l'}$ and renaming of the indices $l \to \beta$, $l' \to l$, we obtain:

$$a_{i\alpha} R^\alpha_{jk\beta} A^\beta_l + a_{j\alpha} R^\alpha_{ik\beta} A^\beta_l = \mathcal{J}_{ij}^{i'j'}(A^\beta_l \lambda_{\beta,i'} g_{j'k} + A^\beta_l \lambda_{\beta,j'} g_{i'k} - A^\beta_l \lambda_{k,i'} g_{j'\beta} - A^\beta_l \lambda_{k,j'} g_{i'\beta}). \tag{16}$$

Because of the symmetries of the curvature tensor,

$$a_{i\alpha} R^\alpha_{jk\beta} A^\beta_l = a^\alpha_i R_{\alpha jk\beta} A^\beta_l = a^\alpha_i R_{\beta kj\alpha} A^\beta_l.$$

Then, equation (16) can be rewritten as

$$a^\alpha_i A^\beta_l R_{\beta kj\alpha} + a^\alpha_j A^\beta_l R_{\beta ki\alpha} = \mathcal{J}_{ij}^{i'j'}(A^\beta_l \lambda_{\beta,i'} g_{j'k} + A^\beta_l \lambda_{\beta,j'} g_{i'k} - A^\beta_l \lambda_{k,i'} g_{j'\beta} - A^\beta_l \lambda_{k,j'} g_{i'\beta}). \tag{17}$$

Symmetrizing with respect to (l,k) and rearranging the terms we obtain

$$
\begin{aligned}
(18)\quad & a_i^\alpha(A_l^\beta R_{\beta kj\alpha} + A_k^\beta R_{\beta lj\alpha}) + a_j^\alpha(A_l^\beta R_{\beta ki\alpha} + A_k^\beta R_{\beta li\alpha}) = \\
&= \mathcal{J}_{ij}^{i'j'}(A_l^\beta \lambda_{\beta,i'} g_{j'k} + A_l^\beta \lambda_{\beta,j'} g_{i'k} - A_l^\beta \lambda_{k,i'} g_{j'\beta} - A_l^\beta \lambda_{k,j'} g_{i'\beta} + \\
&\qquad + A_k^\beta \lambda_{\beta,i'} g_{j'l} + A_k^\beta \lambda_{\beta,j'} g_{i'l} - A_k^\beta \lambda_{l,i'} g_{j'\beta} - A_k^\beta \lambda_{l,j'} g_{i'\beta}).
\end{aligned}
$$

The terms in the brackets in the left hand side are the left hand side of (12.A) with renamed indices: the rules for renaming indices are

$$
\begin{pmatrix} i & \alpha & j & k & l \\ l & \beta & k & j & \alpha \end{pmatrix} \text{ and } \begin{pmatrix} i & \alpha & j & k & l \\ l & \beta & k & i & \alpha \end{pmatrix},
$$

respectively. Substituting (12.A) in (18), we obtain

$$
\begin{aligned}
(19)\quad & \mathcal{J}_{kl}^{k'l'}\left[a_i^\alpha(\Lambda_{\alpha,l'} g_{k'j} + \Lambda_{\alpha,k'} g_{l'j} - \Lambda_{j,l'} g_{k'\alpha} - \Lambda_{j,k'} g_{l'\alpha}) + \right.\\
&\qquad \left. + a_j^\alpha(\Lambda_{\alpha,l'} g_{k'i} + \Lambda_{\alpha,k'} g_{l'i} - \Lambda_{i,l'} g_{k'\alpha} - \Lambda_{i,k'} g_{l'\alpha})\right] = \\
&= \mathcal{J}_{ij}^{i'j'}\left[A_l^\alpha(\lambda_{\alpha,i'} g_{j'k} + \lambda_{\alpha,j'} g_{i'k} - \lambda_{k,i'} g_{j'\alpha} - \lambda_{k,j'} g_{i'\alpha}) + \right.\\
&\qquad \left. + A_k^\alpha(\lambda_{\alpha,i'} g_{j'l} + \lambda_{\alpha,j'} g_{i'l} - \lambda_{l,i'} g_{j'\alpha} - \lambda_{l,j'} g_{i'\alpha})\right].
\end{aligned}
$$

Now we want to change the contraction with the tensor $\mathcal{J}_{kl}^{k'l'}$ by the contraction with the tensor $\mathcal{J}_{ij}^{i'j'}$. This operation is possible (= after applying it we obtain the same equation), because of specific symmetries of each component in brackets. Indeed, for the first component we have

$$
\begin{aligned}
(20)\quad & \mathcal{J}_{kl}^{k'l'} a_i^\alpha \Lambda_{\alpha,l'} g_{k'j} = (\delta_k^{k'}\delta_l^{l'} + J^{k'}{}_k J^{l'}{}_l) a_i^\alpha \Lambda_{\alpha,l'} g_{k'j} = \\
&= \delta_k^{k'}\delta_l^{l'} a_i^\alpha \Lambda_{\alpha,l'} g_{k'j} + J^{k'}{}_k J^{l'}{}_l a_i^\alpha \Lambda_{\alpha,l'} g_{k'j} = \\
&= \delta_i^{i'}\delta_j^{j'} a_{i'}^\alpha \Lambda_{\alpha,l} g_{kj'} + J^{k'}{}_k J^{l'}{}_l a_i^\alpha \Lambda_{\alpha,l'} g_{k'j}
\end{aligned}
$$

Consider the last part and apply Lemma 1 several times:

$$
\begin{aligned}
(21)\quad & J^{k'}{}_k J^{l'}{}_l a_i^\alpha \Lambda_{\alpha,l'} g_{k'j} = (J^{k'}{}_k g_{k'j}) \cdot (J^{l'}{}_l \Lambda_{l',\alpha}) \cdot (g^{\alpha\beta} a_{\beta i}) = \\
&= (-J^{j'}{}_j g_{j'k}) \cdot (-J^{\alpha'}{}_\alpha \Lambda_{\alpha',l}) \cdot (g^{\alpha\beta} a_{\beta i}) = (J^{j'}{}_j g_{j'k}) \cdot \Lambda_{\alpha',l} \cdot (J^{\alpha'}{}_\alpha g^{\alpha\beta}) \cdot a_{\beta i} = \\
&= (J^{j'}{}_j g_{j'k}) \cdot \Lambda_{\alpha',l} \cdot (-J^\beta{}_{\beta'} g^{\beta'\alpha'}) \cdot a_{\beta i} = (J^{j'}{}_j g_{j'k}) \cdot \Lambda_{\alpha',l} g^{\beta'\alpha'} \cdot (-J^\beta{}_{\beta'} a_{\beta i}) = \\
&= (J^{j'}{}_j g_{j'k}) \cdot \Lambda_{\alpha',l} g^{\beta'\alpha'} \cdot (J^{i'}{}_i a_{i'\beta'}) = J^{i'}{}_i J^{j'}{}_j a_{i'}^\alpha \Lambda_{\alpha,l} g_{j'k}
\end{aligned}
$$

Then $\mathcal{J}_{kl}^{k'l'} a_i^\alpha \Lambda_{\alpha,l'} g_{k'j} = \mathcal{J}_{ij}^{i'j'} a_{i'}^\alpha \Lambda_{\alpha,l} g_{kj'}$, as we claimed.

The proof for all other components is analogous (in fact, in the proof we used the hermitian property of $a_{ij}, \lambda_{i,j}, g_{ij}$ only, and this property is fulfilled for all these tensors by Lemma 1).

Therefore, considering each component in the left part of (19) separately, we obtain:

$$
\begin{aligned}
(22)\quad & \mathcal{J}_{ij}^{i'j'}\left[a_{i'}^\alpha(\Lambda_{\alpha,l} g_{kj'} + \Lambda_{\alpha,k} g_{lj'} - \Lambda_{j',l} g_{k\alpha} - \Lambda_{j',k} g_{l\alpha}) + \right.\\
&\qquad \left. + a_{j'}^\alpha(\Lambda_{\alpha,l} g_{ki'} + \Lambda_{\alpha,k} g_{li'} - \Lambda_{i',l} g_{k\alpha} - \Lambda_{i',k} g_{l\alpha})\right] = \\
&= \mathcal{J}_{ij}^{i'j'}\left[A_l^\alpha(\lambda_{\alpha,i'} g_{j'k} + \lambda_{\alpha,j'} g_{i'k} - \lambda_{k,i'} g_{j'\alpha} - \lambda_{k,j'} g_{i'\alpha}) + \right.\\
&\qquad \left. + A_k^\alpha(\lambda_{\alpha,i'} g_{j'l} + \lambda_{\alpha,j'} g_{i'l} - \lambda_{l,i'} g_{j'\alpha} - \lambda_{l,j'} g_{i'\alpha})\right].
\end{aligned}
$$

In the left hand side of (22), we collect the components on containing g with the same indices:

$$
\begin{aligned}
(23)\quad & \mathcal{J}_{ij}^{i'j'}\left[(a_{i'}^\alpha \Lambda_{\alpha,l} - A_l^\alpha \lambda_{\alpha,i'}) g_{kj'} + (a_{i'}^\alpha \Lambda_{\alpha,k} - A_k^\alpha \lambda_{\alpha,i'}) g_{lj'} + \right.\\
&\qquad \left. + (a_{j'}^\alpha \Lambda_{\alpha,l} - A_l^\alpha \lambda_{\alpha,j'}) g_{ki'} + (a_{j'}^\alpha \Lambda_{\alpha,k} - A_k^\alpha \lambda_{\alpha,j'}) g_{li'}\right] = \\
&= \mathcal{J}_{ij}^{i'j'}\left[a_{i'k}\Lambda_{j',l} + a_{i'l}\Lambda_{j',k} + a_{j'k}\Lambda_{i',l} + a_{j'l}\Lambda_{i',k} - A_{j'l}\lambda_{k,i'} - A_{i'l}\lambda_{k,j'} - A_{j'k}\lambda_{l,i'} - A_{i'k}\lambda_{l,j'}\right]
\end{aligned}
$$

We set $c_{il} = a_i^\alpha \Lambda_{\alpha,l} - A_l^\alpha \lambda_{\alpha,i}$. it is easy to check that c_{il} anticommutes with J: $J^\alpha{}_i c_{\alpha j} = -c_{i\alpha} J^\alpha{}_j$. Then equation (23) takes the form:

$$\begin{aligned}(24)\quad & \mathcal{J}_{ij}^{i'j'}\left[c_{i'l}g_{j'k} + c_{i'k}g_{j'l} + c_{j'l}g_{i'k} + c_{j'k}g_{i'l}\right] = \\ &= \mathcal{J}_{ij}^{i'j'}\left[a_{i'k}\Lambda_{j',l} + a_{i'l}\Lambda_{j',k} + a_{j'k}\Lambda_{i',l} + a_{j'l}\Lambda_{i',k} - A_{j'l}\lambda_{k,i'} - A_{i'l}\lambda_{k,j'} - A_{j'k}\lambda_{l,i'} - A_{i'k}\lambda_{l,j'}\right]\end{aligned}$$

Let us now contract the last equation with g^{jk}. This operation involves the j-index, so we have to make use of the explicit formula (9) for $\mathcal{J}$. After some index manipulations, using the anticommutation- and trace-free-properties of the tensors involved[1] , we obtain:

$$(25)\qquad 2nc_{il} + (c_{jk}g^{jk})g_{il} = 0,$$

which implies $c_{il} = 0$. Since $c_{il} = 0$, the equation (24) reads

(26)

$$\mathcal{J}_{ij}^{i'j'}\left[a_{i'k}\Lambda_{j',l} + a_{i'l}\Lambda_{j',k} + a_{j'k}\Lambda_{i',l} + a_{j'l}\Lambda_{i',k} - A_{j'l}\lambda_{k,i'} - A_{i'l}\lambda_{k,j'} - A_{j'k}\lambda_{l,i'} - A_{i'k}\lambda_{l,j'}\right] = 0$$

Let us now multiply (26) by $\frac{1}{2}\mathcal{J}_{pq}^{jk}$. After rearranging components and renaming indices we can write the equation in a more symmetric way:

$$\begin{aligned}(27)\quad & \frac{1}{2}(\delta_i^{i'}\delta_j^{j'}\delta_k^{k'} + \delta_i^{i'}J^{j'}{}_jJ^{k'}{}_k + J^{i'}{}_iJ^{j'}{}_j\delta_k^{k'} - J^{i'}{}_i\delta_j^{j'}J^{k'}{}_k)\cdot \\ &\qquad \cdot(a_{i'k'}\Lambda_{j',l} + a_{i'l}\Lambda_{j',k'} + a_{j'k'}\Lambda_{i',l} + a_{j'l}\Lambda_{i',k'} - \\ &\qquad\qquad - A_{j'l}\lambda_{k',i'} - A_{i'l}\lambda_{k',j'} - A_{j'k'}\lambda_{l,i'} - A_{i'k'}\lambda_{l,j'}) = 0\end{aligned}$$

Using that J anticommutes with a_{ij}, A_{ij}, $\lambda_{i,j}$ (see Lemma 1) one can get

(28)

$$a_{il}\Lambda_{j,k} + a_{jk}\Lambda_{i,l} + J^{i'}{}_iJ^{j'}{}_j(a_{i'l}\Lambda_{j',k} + a_{j'k}\Lambda_{i',l}) = A_{il}\lambda_{j,k} + A_{jk}\lambda_{i,l} + J^{i'}{}_iJ^{j'}{}_j(A_{i'l}\lambda_{j',k} + A_{j'k}\lambda_{i',l})$$

Symmetrizing (28) by (i,l) we finally obtain

$$(29)\qquad a_{il}\Lambda_{j,k} + a_{jk}\Lambda_{i,l} = A_{il}\lambda_{j,k} + A_{jk}\lambda_{i,l}.$$

In other words, $\Lambda_\aleph a_\daleth + \Lambda_\daleth a_\aleph = \lambda_\daleth A_\aleph + \lambda_\aleph A_\daleth$, where $\aleph$ and $\daleth$ stand for the symmetric indices jl and ik, respectively.

But it is easy to check that a non-zero simple symmetric tensor $X_{\aleph\daleth} = P_\aleph Q_\daleth + P_\daleth Q_\aleph$ determines its factors $P_\aleph$ and $Q_\daleth$ up to scale and order (it is sufficient to check, for example, by taking $P_\aleph$ and $Q_\daleth$ to be basis vectors). Since a_{ij} and A_{ij} are supposed to be linearly independent, it follows that $\lambda_{i,j} = \mathrm{const}\cdot a_{ij}$, as required. □

Remark 3. We would like to emphasize here that, though Lemma 2 is formulated in the differential-geometrical notation, it is essentially an algebraic statement (in the proof we did not use differentiation except for the integrability conditions (12) that were actually obtained before, see (7)). Moreover, we can replace R^i_{jkl} in (12) by any $(1,3)$-tensor having the same algebraic symmetries (with respect to g) as the curvature tensor.

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[1]Each component separately:

$$c_{il}g_{jk}g^{jk} = 2nc_{il}, \quad c_{ik}g_{jl}g^{jk} = c_{il},$$
$$c_{jl}g_{ik}g^{jk} = c_{il}, \quad c_{jk}g_{il}g^{jk} = (c_{jk}g^{jk})g_{il},$$
$$J^{i'}{}_iJ^{j'}{}_jc_{i'l}g_{j'k}g^{jk} = 0, \quad J^{i'}{}_iJ^{j'}{}_jc_{i'k}g_{j'l}g^{jk} = -c_{il},$$
$$J^{i'}{}_iJ^{j'}{}_jc_{j'l}g_{i'k}g^{jk} = -J^{i'}{}_iJ^{l'}{}_lc_{i'l'} = -c_{il}, \quad J^{i'}{}_iJ^{j'}{}_jc_{j'k}g_{i'l}g^{jk} = -(g^{jk}J^{j'}{}_jc_{j'k})J^{i'}{}_ig_{i'l} = 0,$$
$$a_{ik}\Lambda_{j,l}g^{jk} = a_i^p\Lambda_{p,l}, \quad a_{il}\Lambda_{j,k}g^{jk} = 0, \quad a_{jk}\Lambda_{i,l}g^{jk} = 0, \quad a_{jl}\Lambda_{i,k}g^{jk} = \Lambda_{i,p}a_l^p,$$
$$J^{i'}{}_iJ^{j'}{}_ja_{i'k}\Lambda_{j',l}g^{jk} = -a_i^p\Lambda_{p,l}, \quad J^{i'}{}_iJ^{j'}{}_ja_{i'l}\Lambda_{j',k}g^{jk} = 0, \quad J^{i'}{}_iJ^{j'}{}_ja_{j'k}\Lambda_{i',l}g^{jk} = 0, \quad J^{i'}{}_iJ^{j'}{}_ja_{j'l}\Lambda_{i',k}g^{jk} = -\Lambda_{i,p}a_l^p.$$

2.3. **If the solutions a_{ij}, A_{ij} and g_{ij} are linearly dependent over functions, then they are linearly dependent over constants.** The goal of this section is to show, that under the assumption of degree of mobility ≥ 3, equation (11) holds in a neighborhood of almost every point of M for each solution (a_{ij}, λ_i) of equation (3). The real numbers B and μ in equation (11) then become smooth function on this neighborhood. In the end of this section, it will be also shown that the local function B is the same for all solutions of equation (3).

Lemma 3. *On a Kähler manifold $(M^{2n\geq 4}, g, J)$, let (A_{ij}, λ_i) and (a_{ij}, λ_i) be solutions of (3). Then, almost every point $p \in M$ has a neighborhood $U(p) \ni p$ such that in this neighborhood one of the following conditions is fulfilled:*

(a) *a_{ij}, A_{ij}, and g_{ij} are linearly independent at every point of $U(p)$,*
(b) *a_{ij}, A_{ij}, and g_{ij} are linearly dependent at every point of $U(p)$.*

Proof. The proof in fact does not require that a_{ij} and A_{ij} are solutions of (3). Let W be the set of the points where (a) is fulfilled. W is evidently an open set. Consider $\operatorname{int}(M \setminus W)$, where "int" denotes the set of the interior points. This is also an open set, and $W \cup \operatorname{int}(M \setminus W)$ is open and everywhere dense. By construction, every point of $W \cup \operatorname{int}(M \setminus W)$ has a neighborhood satisfying the condition (a) or the condition (b). □

One of the possibilities in Lemma 3 is that (in a neighborhood $U(p)$) the solutions a_{ij}, A_{ij} and g_{ij} of (3) are linearly depended over functions. Our goal is to show that in this case they are actually linearly dependent (over constants). At first we consider the special case, when two solutions are proportional.

Lemma 4. *Let (M^{2n}, g, J) be a Kähler manifold of dimension $2n \geq 4$, and let (a_{ij}, λ_i) and (A_{ij}, Λ_i) be solutions of (3) such that $a_{ij} \neq 0$ at every point of some open subset $U \subseteq M$. If $\alpha : U \to \mathbb{R}$ is a function such that*

$$A = \alpha\, a, \tag{30}$$

then α is constant, and $A = \alpha\, a$ on the whole M.

Proof. Since A_{ij} and a_{ij} are smooth tensor fields on U and $a_{ij} \neq 0$, the function α is also smooth. We covariantly differentiate (30) and substitute the derivatives of a_{ij} and A_{ij} using (3) to obtain

$$\gamma_i g_{jk} + \gamma_j g_{ik} - \bar{\gamma}_i J_{jk} - \bar{\gamma}_j J_{ik} = \alpha_{,k} a_{ij}, \tag{31}$$

where $\gamma_i := \Lambda_i - \alpha\lambda_i$. Contracting equation (31) with a non-zero vector field U^k such that $U^k \alpha_{,k} = 0$ yields

$$\gamma_i U_j + \gamma_j U_i + \bar{\gamma}_i \bar{U}_j + \bar{\gamma}_j \bar{U}_i = 0 \tag{32}$$

Let us now show that at every point

$$\operatorname{span}\{U^j, \bar{U}^j\}^\perp \subseteq \operatorname{span}\{\gamma^j, \bar{\gamma}^j\}^\perp.$$

For every vector field $V^j \in \operatorname{span}\{U^j, \bar{U}^j\}^\perp$ we have (contracting this vector field with (32))

$$(\gamma_j V^j)\, U_i + (\bar{\gamma}_j V^j)\, \bar{U}_i = 0$$

Since U_i and $\bar{U}_i$ are linearly independent, $\gamma_j V^j = \bar{\gamma}_j V^j = 0$. Then $V_i \in \operatorname{span}\{\gamma^j, \bar{\gamma}^j\}^\perp$. Thus, $\operatorname{span}\{U^j, \bar{U}^j\}^\perp \subseteq \operatorname{span}\{\gamma^j, \bar{\gamma}^j\}^\perp$ as we claimed.

Assume $\gamma_i \neq 0$. Then the spaces $\operatorname{span}\{U^j, \bar{U}^j\}^\perp$ and $\operatorname{span}\{\gamma^j, \bar{\gamma}^j\}^\perp$ have equal dimension $(2n-2)$, and therefore coincide. The same holds for their orthogonal complements and we obtain

$$\operatorname{span}\{U^j, \bar{U}^j\} = \operatorname{span}\{\gamma^j, \bar{\gamma}^j\}$$

Thus, every vector U^i from the at least $(2n-1)$-dimensional space $\operatorname{span}(\alpha_{,}{}^{i})^\perp$ lies in the 2-dimensional space $\operatorname{span}\{\gamma^j, \bar{\gamma}^j\}$, which gives us a contradiction. Thus, $\gamma_i = 0$ and equation (31) reads $\alpha_{,k} a_{ij} = 0$, implying α is constant on U. Therefore, the solution $A_{ij} - \alpha a_{ij}$ vanishes at every point of U. By Corollary 4 it vanishes on the whole M. □

Now let us treat the general case:

Lemma 5. *On a connected Kähler manifold (M^{2n}, g, J) of dimension $2n \geq 4$, let (a_{ij}, λ_i) and (A_{ij}, Λ_i) be solutions of (3). Assume that for certain functions α and β on an open subset $U \subseteq M$ we have*

$$A_{ij} = \alpha g_{ij} + \beta a_{ij} \tag{33}$$

Then there exist constants $(C_1, C_2, C_3) \neq (0, 0, 0)$ such that

$$C_1 A + C_2 a + C_3 g = 0 \quad \text{on the whole } M.$$

Proof. If there locally exists a function c such that $a_{ij} = c g_{ij}$, then by the previous Lemma 4 the function c is a constant. Hence, by Corollary 4, one can choose $C_1 = 0$, $C_2 = -1$ and $C_3 = c$.

Let a_{ij} be non-proportional to g_{ij}. Then (33) is a linear system of equations of maximal rank with smooth coefficients on functions α and β. Thus, its solutions α and β are smooth.

Similarly as before in Lemma 4, by differentiating (33) we obtain

$$\gamma_i g_{jk} + \gamma_j g_{ik} - \bar{\gamma}_i J_{jk} - \bar{\gamma}_j J_{ik} = \alpha_{,k} g_{ij} + \beta_{,k} a_{ij} \tag{34}$$

where $\gamma_i = \Lambda_i - \beta \lambda_i$.

Assume $\gamma_i \neq 0$. We contract (34) with a vector field U^i such that $U^k \alpha_{,k} = U^k \beta_{,k} = 0$ to obtain equation (32). As in the proof of Lemma 4, we obtain

$$\operatorname{span}\{U^j, \bar{U}^j\} = \operatorname{span}\{\gamma^j, \bar{\gamma}^j\}$$

implying $U_i = c \cdot \gamma_i + d \cdot \bar{\gamma}_i$ for certain functions c and d. We substitute U_i in (32) to obtain

$$2c \cdot (\gamma_i \gamma_j + \bar{\gamma}_i \bar{\gamma}_j) = 0.$$

Since $\gamma_i \neq 0$, it follows that $c = 0$, and therefore $U^i = d \cdot \bar{\gamma}^i$. We have shown that every vector U^i from the at least $(2n-2)$-dimensional space $\operatorname{span}(\alpha_,{}^i, \beta_,{}^i)^\perp$ is proportional to $\bar{\gamma}^i$, which gives us a contradiction. Thus, $\gamma_i = 0$ and equation (34) takes the form

$$\alpha_{,k} a_{ij} + \beta_{,k} g_{ij} = 0.$$

We have $\alpha_{,k} \equiv 0 \equiv \beta_{,k}$, implying $\alpha \equiv \text{const} =: C_2$ and $\beta = \text{const} =: C_3$.

Therefore, the solution $A_{ij} - C_2 a_{ij} - C_3 g_{ij}$ vanishes at every point of U. By Corollary 4 it vanishes on the whole M. □

Thus, if the degree of mobility is ≥ 3, by Lemma 5, for every solution (a_{ij}, λ_i) of (3) such that $a_{ij} \neq \text{const} \cdot g_{ij}$, equation (11) holds in a neighborhood of almost every point of M (for some locally defined functions B and μ that could a priori depend on the solution (a_{ij}, λ_i)). Our next goal is to show, that the function B is the same for all solutions:

Corollary 5. *Let (M^{2n}, g, J) be a Kähler manifold of dimension $2n \geq 4$ and assume that the degree of mobility is ≥ 3. Then, the function B defined by equation (11) does not depend on the solution (a_{ij}, λ_i) of equation (3).*

Proof. Take the second solution (A_{ij}, Λ_i) of equation (3). Let us first assume that g_{ij}, a_{ij} and A_{ij} are linearly independent.

We know that $(a_{ij} + A_{ij}, \lambda_i + \Lambda_i)$ is again a solution. Adding equations (11) for (a_{ij}, λ_i) and (A_{ij}, Λ_i) with functions B and B' respectively and substracting the same equation corresponding to the sum of the both solutions (the correspondent function B for the sum of solutions will be denoted by B^+), we obtain

$$0 = \text{something} \cdot g_{ij} + (B - B^+) a_{ij} + (B' - B^+) A_{ij}$$

Combining Lemma 5 and the assumption that g, a and A are linearly independent, we obtain $B = B^+ = B'$ as we claimed.

Consider now the second case when g_{ij}, a_{ij} and A_{ij} are linearly dependent, i.e. (without loss of generality), $A_{ij} = C g_{ij} + D a_{ij}$ on M for some constants C and D. Thus, the corresponding

1-forms Λ_i and λ_i for A_{ij} and a_{ij} respectively are related by the equation $\Lambda_i = D\lambda_i$. Multiplying equation (11) by D we obtain

$$\underbrace{D\lambda_{i,j}}_{\Lambda_{i,j}} = D\mu g_{ij} + DBa_{ij} = \underbrace{(D\mu - CB)}_{\mathcal{M}} g_{ij} + \underbrace{(Da_{ij} + Cg_{ij})}_{A_{ij}} B \tag{35}$$

This is equation (11) on (A_{ij}, Λ_i) with the same function B. Finally, in all cases, the function B is the same for all solutions of equation (3). □

2.4. **In the neighborhood of a point such that g, a, and A are linearly independent, the function B is a constant.** Our next goal is to show that the local function B we have found is a constant.

Lemma 6. *Let (M^{2n}, g, J) be a Kähler manifold of dimension $2n \geq 4$. Suppose that in a neighborhood $U \subseteq M$ there exist at least two solutions (a_{ij}, λ_i) and (A_{ij}, Λ_i) of (3) such that a, A and g are linearly independent at every point of U. Then the function B defined by equation (11) is a constant.*

The proofs for the cases $\dim M \geq 6$ and $\dim M = 4$ use different methods and will be given in sections 2.4.1 and 2.4.2 respectively.

2.4.1. *Proof of Lemma 6, if* $\dim M \geq 6$. First of all, the function B is smooth. Indeed, the trace-free version of (11) is

$$\lambda_{i,j} - \tfrac{1}{2n}\lambda_{k,}{}^{k} \cdot g_{ij} = B(a_{ij} - \tfrac{2}{n}\lambda g_{ij}), \tag{36}$$

where $\lambda := \frac{1}{4}a^i_i$, and the function B is smooth since it is the coefficient of the proportionality of the nowhere vanishing tensor $(a_{ij} - \frac{2}{n}\lambda g_{ij})$ and the tensor $(\lambda_{i,j} - \frac{1}{2n}\lambda_{k,}{}^{k} \cdot g_{ij})$. Since B is smooth, μ is smooth as well, as the coefficient of the proportionality of the nowhere vanishing tensor g_{ij} and the tensor $(\lambda_{i,j} - Ba_{ij})$.

Thus, all objects in the equation

$$\lambda_{i,j} = \mu g_{ij} + Ba_{ij} \tag{37}$$

are smooth. We covariantly differentiate the equation and substitute $a_{ij,k}$ using (10) to obtain

$$\lambda_{i,jk} = \mu_{,k}g_{ij} + B_{,k}a_{ij} + Ba_{ij,k} = \mu_{,k}g_{ij} + B_{,k}a_{ij} + B\cdot \mathcal{J}^{i'j'}_{ij}(\lambda_{i'}g_{j'k} + \lambda_{j'}g_{i'k}). \tag{38}$$

By definition of the curvature tensor,

$$\begin{aligned}\lambda_p R^p_{ijk} = \lambda_{i,jk} - \lambda_{i,kj} &\overset{(38)}{=} \mu_{,k}g_{ij} - \mu_{,j}g_{ik} + B_{,k}a_{ij} - B_{,j}a_{ik} + \\ &+ B\cdot\mathcal{J}^{i'j'}_{ij}(\lambda_{i'}g_{j'k} + \lambda_{j'}g_{i'k}) - B\cdot\mathcal{J}^{i'k'}_{ik}(\lambda_{i'}g_{k'j} + \lambda_{k'}g_{i'j}) = \\ &= \mu_{,k}g_{ij} - \mu_{,j}g_{ik} + B_{,k}a_{ij} - B_{,j}a_{ik} + B\lambda_j g_{ik} - B\lambda_k g_{ij} + \\ &\qquad + B\cdot J^{i'}{}_i J^{j'}{}_j(2\lambda_{i'}g_{j'k} + \lambda_{j'}g_{i'k}) - BJ^{i'}{}_i J^{k'}{}_k \lambda_{k'}g_{i'j}\end{aligned} \tag{39}$$

Let us now substitute $\lambda_{i,j}$ in (12) by (37). The components with μ disappear because of the symmetries of g_{ij} and the equation takes the following form:

$$a_{i\alpha}R^\alpha_{jkl} + a_{j\alpha}R^\alpha_{ikl} = B\mathcal{J}^{i'j'}_{ij}(a_{li'}g_{j'k} + a_{lj'}g_{i'k} - a_{ki'}g_{j'l} - a_{kj'}g_{i'l}) \tag{40}$$

We contract this equation with λ^l. Applying the identity $a_{i\alpha}R^\alpha_{jk\beta}\lambda^\beta = a^\alpha_i\lambda_\beta R^\beta_{kj\alpha}$ we obtain

$$a^\alpha_i\lambda_\beta R^\beta_{kj\alpha} + a^\alpha_j\lambda_\beta R^\beta_{ki\alpha} = B\mathcal{J}^{i'j'}_{ij}(\lambda^\beta a_{\beta i'}g_{j'k} + \lambda^\beta a_{\beta j'}g_{i'k} - a_{ki'}\lambda_{j'} - a_{kj'}\lambda_{i'}). \tag{41}$$

Now we substitute the left hand side using (39). After substituting (9) for $\mathcal{J}^{i'j'}_{ij}$ and tensor manipulation, we obtain

$$\begin{aligned}&g_{kj}(a^\alpha_i\mu_{,\alpha} - 2B\lambda^\alpha a_{i\alpha}) + g_{ki}(a^\alpha_j\mu_{,\alpha} - 2B\lambda^\alpha a_{j\alpha}) + \\ &\qquad + a_{kj}(a^\alpha_i B_{,\alpha} - \mu_{,i} + 2B\lambda_i) + a_{ki}(a^\alpha_j B_{,\alpha} - \mu_{,j} + 2B\lambda_j) = \\ &\qquad\qquad = B_{,j}a_{k\alpha}a^\alpha_i + B_{,i}a_{k\alpha}a^\alpha_j\end{aligned} \tag{42}$$

Set $\xi_i := a_i^\alpha \mu_{,\alpha} - 2B\lambda^\alpha a_{i\alpha}$ and $\eta_i := a_i^\alpha B_{,\alpha} - \mu_{,i} + 2B\lambda_i$. Then

$$\xi_i g_{kj} + \xi_j g_{ki} + \eta_i a_{kj} + \eta_j a_{ki} = B_{,j} a_{k\alpha} a_i^\alpha + B_{,i} a_{k\alpha} a_j^\alpha \tag{43}$$

Remark 4. For further use let us note that if $B = \text{const}$, i.e., if $B_{,i} \equiv 0$, then the right-hand side of the last equation vanishes implying $\eta_i \equiv 0$. Then,

$$\mu_{,i} = 2B\lambda_i. \tag{44}$$

Let us now alternate (43) with respect to (i,k), rename $j \longleftrightarrow k$ and add the result to (43). After this manipulation only the terms that are symmetric with respect to (j,k) remain, and we obtain

$$\xi_i g_{jk} + \eta_i a_{jk} = B_{,i} a_{k\alpha} a_j^\alpha \tag{45}$$

If $B_{,i} \neq 0$, equation (45) implies that for certain functions C and D

$$C g_{jk} + D a_{jk} = a_{k\alpha} a_j^\alpha \tag{46}$$

Let us now calculate $\nabla_k (a_{i\alpha} a_j^\alpha)$:

$$\begin{aligned}\nabla_k(a_{i\alpha}a_j^\alpha) &= a_{i\alpha,k}a_j^\alpha + a_{j\alpha,k}a_i^\alpha = \\ &= \mathcal{J}_{ij}^{i'j'}(\lambda_{i'}a_{j'k} + \lambda_{j'}a_{i'k} + \lambda_\alpha a_{i'}^\alpha g_{j'k} + \lambda_\alpha a_{j'}^\alpha g_{i'k}) \overset{(46)}{=} \\ &= C_{,k}g_{ij} + D_{,k}a_{ij} + D\mathcal{J}_{ij}^{i'j'}(\lambda_{i'}g_{j'k} + \lambda_{j'}g_{i'k})\end{aligned} \tag{47}$$

Setting $s_i := \lambda_\alpha a_{i'}^\alpha - D\lambda_i$, we obtain

$$\mathcal{J}_{ij}^{i'j'}(\lambda_{i'}a_{j'k} + \lambda_{j'}a_{i'k} + s_i g_{j'k} + s_j g_{i'k}) - C_{,k}g_{i'j'} - D_{,k}a_{i'j'} = 0 \tag{48}$$

To simplify this equation consider the action of the operator $\mathcal{J}_{kj}^{k'j'}$ on it. After applying the properties of the complex structure, the equation takes the form

$$\mathcal{J}_{ij}^{i'j'}\left(\lambda_{i'}a_{j'k} + s_{i'}g_{j'k}\right) - \mathcal{J}_{kj}^{k'j'}\left(\frac{C_{,k'}}{2}g_{ij'} + \frac{D_{,k'}}{2}a_{ij'}\right) = 0. \tag{49}$$

Alternating with respect to (i,k) and collecting the terms yields

(50)

$$\mathcal{J}_{ij}^{i'j'}\left[a_{j'k}\left(\lambda_{i'} + \frac{D_{,i'}}{2}\right) + g_{j'k}\left(s_{i'} + \frac{C_{,i'}}{2}\right)\right] - \mathcal{J}_{kj}^{k'j'}\left[a_{ij'}\left(\lambda_{k'} + \frac{D_{,k'}}{2}\right) + g_{ij'}\left(s_{i'} + \frac{C_{,i'}}{2}\right)\right] = 0$$

After denoting

$$\tau_i = s_i + \frac{C_{,i}}{2}, \quad \bar\tau_i = J^{i'}{}_i\tau_{i'}, \quad g_{j\bar k} = J^{k'}{}_k g_{jk'} \tag{51}$$

$$\nu_i = \lambda_i + \frac{D_{,i}}{2}, \quad \bar\nu_i = J^{i'}{}_i\nu_{i'}, \quad a_{j\bar k} = J^{k'}{}_k a_{jk'}, \tag{52}$$

equation (50) reads

$$(\tau_i g_{jk} - \tau_k g_{ij}) - (\bar\tau_i g_{j\bar k} - \bar\tau_k g_{j\bar i}) + (\nu_i a_{jk} - \nu_k a_{ij}) - (\bar\nu_i a_{j\bar k} - \bar\nu_k a_{j\bar i}) = 0 \tag{53}$$

Let us now contract this equation with a certain vector field ξ^j. We obtain

$$(\tau_i\xi_k - \tau_k\xi_i) - (\bar\tau_i\bar\xi_k - \bar\tau_k\bar\xi_i) = (\nu_i\eta_k - \nu_k\eta_i) - (\bar\nu_i\bar\eta_k - \bar\nu_k\bar\eta_i) \tag{54}$$

where $\bar\xi_i = J^{i'}{}_i\xi_{i'}$, $\eta_i = -a_{ij}\xi^j$ and $\bar\eta_i = J^{i'}{}_i\eta_{i'}$.

If the vectors τ_i, ξ_i, $\bar\tau_i$ and $\bar\xi_i$ are linearly independent, this equation implies that the 4-dimensional space $l(\tau,\xi)$ spanned over $\{\tau_i, \xi_i, \bar\tau_i, \bar\xi_i\}$ coincides with $l(\nu,\eta)$ spanned over $\{\nu_i, \eta_i, \bar\nu_i, \bar\eta_i\}$. Indeed, these spaces are determined as the orthogonal complements to the kernels of the corresponding 2-forms

$$Ker(\tau,\xi) = \{u^i \mid \left((\tau_i\xi_k - \tau_k\xi_i) - (\bar\tau_i\bar\xi_k - \bar\tau_k\bar\xi_i)\right) u^i x^k = 0 \text{ for every } x^k\}$$
$$Ker(\nu,\eta) = \{u^i \mid ((\nu_i\eta_k - \nu_k\eta_i) - (\bar\nu_i\bar\eta_k - \bar\nu_k\bar\eta_i))\, u^i x^k = 0 \text{ for every } x^k\}$$

Since by (54) the forms are equal, the subspaces are equal as well.

If $\dim M \geq 6$, there exist two vectors $\overset{1}{\xi}{}^j$ and $\overset{2}{\xi}{}^j$ such that $\{\tau_i, \overset{1}{\xi}_i, \overset{2}{\xi}_i, \bar\tau_i, \overset{1}{\bar\xi}_i, \overset{2}{\bar\xi}_i\}$ are linearly independent. Then $l(\tau,\overset{1}{\xi})$ and $l(\tau,\overset{2}{\xi})$ intersect along the 2-dimensional subspace spanned by the vectors $\{\tau_i,\bar\tau_i\}$. The corresponding vectors $\overset{1}{\eta}$ and $\overset{2}{\eta}$ determine spaces $l(\nu,\overset{1}{\eta})$ and $l(\nu,\overset{2}{\eta})$ which intersect along the subspace spanned by the vectors $\{\nu_i,\bar\nu_i\}$). Since the 4-dimensional spaces are pairwise equal, one obtains

$$span\{\tau_i,\bar\tau_i\} = span\{\nu_i,\bar\nu_i\} \tag{55}$$

Then, for certain functions p, q we have

$$\tau_i = p\nu_i + q\bar\nu_i, \quad \bar\tau_i = p\bar\nu_i - q\nu_i.$$

Let us now substitute this in (53). After collecting terms, we obtain

$$\begin{multline} \nu_i(p\, g_{jk} + q\, J^{k'}{}_k g_{jk'} + a_{jk}) - \nu_k(p\, g_{ij} + q\, J^{i'}{}_i g_{i'j} + a_{ij}) = \\ \bar\nu_i(p\, J^{k'}{}_k g_{jk'} - q\, g_{jk} + J^{k'}{}_k a_{jk}) - \bar\nu_k(p\, J^{i'}{}_i g_{i'j} - q\, g_{ij} + J^{i'}{}_i a_{i'j}). \end{multline} \tag{56}$$

Defining

$$\omega_{jk} = p\, g_{jk} + q\, J^{k'}{}_k g_{jk'} + a_{jk}, \tag{57}$$
$$\omega_{j\bar k} = p\, J^{k'}{}_k g_{jk'} - q\, g_{jk} + J^{k'}{}_k a_{jk}, \tag{58}$$

we can rewrite equation (56) in the form

$$\nu_i\omega_{jk} - \nu_k\omega_{ji} = \bar\nu_i\omega_{j\bar k} - \bar\nu_k\omega_{j\bar\imath} \tag{59}$$

This equation has the same structure as (53), but with a non-symmetric, hermitian bilinear form ω_{jk}. One can easily see that it holds if and only if

$$\omega_{jk} = \alpha_j\nu_k + J^{j'}{}_j J^{k'}{}_k \alpha_{j'}\nu_{k'} \tag{60}$$

for some covector α_j.

Substituting ω in (57) and alternating the result, we obtain

$$2qJ^{k'}{}_k g_{jk'} = \alpha_j\nu_k - \alpha_k\nu_j + J^{j'}{}_j J^{k'}{}_k(\alpha_{j'}\nu_{k'} - \alpha_{k'}\nu_{j'}).$$

Let us now consider this equation as an equality between two bilinear forms. The rank of the right-hand side is not greater then 4, while the left hand side is nondegenerate unless $q \neq 0$. Since $\dim M \geq 6$ we have $q = 0$ and ω_{jk} is symmetric by (57). Thus,

$$\omega_{jk} = \alpha(\nu_j\nu_k + J^{j'}{}_j J^{k'}{}_k \nu_{j'}\nu_{k'}), \tag{61}$$

where α is a scalar function. It immediately follows that (after renaming of variables)

$$a_{ij} = p(u_i u_j + J^{i'}{}_i J^{j'}{}_j u_{i'} u_{j'}) + q g_{ij}, \tag{62}$$

where p, q — are certain functions and u_i is a covariant vector field.

We have shown that if in a neighborhood of some point there are two linearly independent solutions of the extended system with non-constant B, then each solution has the special form (62).

Now we would like to show that the function q, corresponding to a solution a_{ij} as was given in equation (62), is a constant. In order to do this, take an arbitrary $U^i \in \operatorname{span}\{u^i, \bar{u}^i\}^\perp$. Contracting (62) with U^i we see that

$$a_{i\alpha}U^\alpha = qU_i$$

Hence all vectors, orthogonal to u and $\bar{u}$, correspond to the eigenvalue q of $a^i_j = g^{i\alpha}a_{\alpha j}$. Taking the derivative of the equation above and inserting equation (3) yields

$$\lambda_i U_k + \lambda_\alpha U^\alpha g_{ik} - \bar{\lambda}_i \bar{U}_k - \bar{\lambda}_\alpha U^\alpha J_{ik} + a_{i\alpha}U^\alpha_{,k} = q_{,k}U_i + qU_{i,k}$$

Contracting this equation with U^i gives

$$2\lambda_\alpha U^\alpha U_k - 2\bar{\lambda}_\alpha U^\alpha \bar{U}_k = q_{,k}U_\alpha U^\alpha. \tag{63}$$

Thus, $q_{,k} \in \operatorname{span}\{U_k, \bar{U}_k\}$ unless $U_\alpha U^\alpha = 0$.

Given any vector $U^i \in \operatorname{span}\{u, \bar{u}\}^\perp$, such that $U_\alpha U^\alpha \neq 0$, we can construct a second vector $W^i \in \operatorname{span}\{u, \bar{u}\}^\perp$ such that $W_\alpha W^\alpha \neq 0$ and $\operatorname{span}\{U^i, \bar{U}^i\} \cap \operatorname{span}\{W^i, \bar{W}^i\} = \{0\}$. In this case, using equation (63) for U^i and W^i, we obtain that q is a constant (because $q_k \in \operatorname{span}\{U_k, \bar{U}_k\} \cap \operatorname{span}\{W_k, \bar{W}_k\} = \{\vec{0}\}$). It remains to show that such a vector U^i exists. Assuming each vector $U^i \in \operatorname{span}\{u, \bar{u}\}^\perp$ satisfies $U_\alpha U^\alpha = 0$, we obtain that $U_\alpha W^\alpha = 0$ for all $U^i, W^i \in \operatorname{span}\{u, \bar{u}\}^\perp$. Since $\dim M \geq 6$, this means that $\dim \operatorname{span}\{u, \bar{u}\} = \dim\left((\operatorname{span}\{u, \bar{u}\})^\perp\right)^\perp \geq 4$ which is a contradiction.

Using that q is a constant, we can substract the trivial solution qg_{ij} from a_{ij} and include the function p in the vector field u_i. In other words, without loss of generality, a_{ij} is given by

$$a_{ij} = u_i u_j + J^{i'}{}_i J^{j'}{}_j u_{i'} u_{j'}.$$

Note that u^j is an eigenvector of a^i_j as well. If the corresponding eigenvalue is a constant, all eigenvalues of a^i_j are constant. Hence, the trace of a^i_j is constant, and the 1-form $\lambda_i = \frac{1}{4}(a^k_k)_{,i}$ is identically zero. Inserting $\lambda_i \equiv 0$ in equation (11), we see that

$$0 = \mu g_{ij} + Ba_{ij}$$

By Lemma 5, $\mu = B = 0$, since g_{ij} and a_{ij} are assumed to be linearly independent. We see that in this case $B = \text{const}$ as we claim.

Now consider the case when the eigenvalue corresponding to the eigenvector u^i is not constant. We obtain that $\operatorname{span}\{\lambda_i, \bar{\lambda}_i\} = \operatorname{span}\{u_i, \bar{u}_i\}$, since λ_i and $\bar{\lambda}_i$ are contained in the sum of the eigenspaces, corresponding to the non-constant eigenvalues. Consider the second solution

$$A_{ij} = v_i v_j + J^{i'}{}_i J^{j'}{}_j v_{i'} v_{j'}$$

of the extended system, such that a_{ij}, A_{ij}, g_{ij} are linearly independent. By Λ_i, we denote the 1-form corresponding to A_{ij}. The sum

$$a_{ij} + A_{ij} = u_i u_j + J^{i'}{}_i J^{j'}{}_j u_{i'} u_{j'} + v_i v_j + J^{i'}{}_i J^{j'}{}_j v_{i'} v_{j'}$$

is again a solution of equation (3) and hence, can be written as

$$a_{ij} + A_{ij} = w_i w_j + J^{i'}{}_i J^{j'}{}_j w_{i'} w_{j'} + Qg_{ij}$$

Comparing the last two equations, we see that

$$Qg_{ij} = u_i u_j + \bar{u}_i \bar{u}_j + v_i v_j + \bar{v}_i \bar{v}_j - w_i w_j - \bar{w}_i \bar{w}_j.$$

Since $\mathrm{span}\{\lambda_i, \bar\lambda_i\} = \mathrm{span}\{u_i, \bar u_i\}, \mathrm{span}\{\Lambda_i, \bar\Lambda_i\} = \mathrm{span}\{v_i, \bar v_i\}$ and $\mathrm{span}\{w_i, \bar w_i\} = \mathrm{span}\{\lambda_i + \Lambda_i, \bar\lambda_i + \bar\Lambda_i\}$, the right-hand side has rank at most 4 and therefore, $Q \equiv 0$. Let us rewrite the last equation in the form

$$w_i w_j + \bar w_i \bar w_j = u_i u_j + \bar u_i \bar u_j + v_i v_j + \bar v_i \bar v_j$$

Since the left hand side has rank 2, u_i, $\bar u_i$, v_i and $\bar v_i$ are linearly dependent and the intersection $\mathrm{span}\{u_i, \bar u_i\} \cap \mathrm{span}\{v_i, \bar v_i\}$ is non-empty. Since it is also J-invariant, we obtain that $\mathrm{span}\{u_i, \bar u_i\} = \mathrm{span}\{v_i, \bar v_i\}$. Thus, $v_i = \alpha u_i + \beta \bar u_i$, for some real constants α, β. It follows, that $\bar v_i = \alpha \bar u_i - \beta u_i$ and we obtain

$$\begin{aligned} v_i v_j + \bar v_i \bar v_j &= (\alpha u_i + \beta \bar u_i)(\alpha u_j + \beta \bar u_j) + (\alpha \bar u_i - \beta u_i)(\alpha \bar u_j - \beta u_j) \\ &= (\alpha^2 + \beta^2)(u_i u_j + \bar u_i \bar u_j) \end{aligned}$$

Inserting this in the original formulas for a_{ij} and A_{ij}, we see that $a_{ij} = \mathrm{const} \cdot A_{ij}$. We obtain a contradiction to the assumption that a_{ij} and A_{ij} are linearly independent. Lemma 6 is proved under the assumption $\dim M \geq 6$.

2.4.2. *Proof of Lemma 6 in case* $\dim M = 4$.

Lemma 7. *Let (M^{2n}, g, J) be a Kähler manifold of dimension $2n = 4$ and assume that the degree of mobility of the metric g is ≥ 3. Then g has constant holomorphic sectional curvature $-4B$, where B is defined by equation (11). In particular, B is a constant.*

Remark 5. As we see, Lemma 7 contains an extra statement: not only $B = \mathrm{const}$, but also the metric g has constant holomorphic sectional curvature. This result was actually unexpected. Indeed, the analog of dimension 4 in the theory of projectively equivalent metrics is 2, and in dimension 2 there exist metrics of non-constant sectional curvature admitting 4-parametric family of projectively equivalent metrics.

Proof. We will work in a small neighborhood of the point $p \in M$, such that there exist three solutions g_{ij}, a_{ij} and A_{ij} of equation (3), linearly independent at p.

Using equation (11), we substitute $\lambda_{i,j}$ in equation (7) to obtain

$$a_{i\alpha} R^{\alpha}_{jkl} + a_{j\alpha} R^{\alpha}_{ikl} = -4B(a_{i\alpha} K^{\alpha}_{jkl} + a_{j\alpha} K^{\alpha}_{ikl}), \tag{64}$$

where K is the algebraic curvature tensor of constant holomorphic sectional curvature equal to 1, namely

$$K^{\alpha}_{jkl} = \frac{1}{4}(\delta^{\alpha}_k g_{jl} - \delta^{\alpha}_l g_{jk} + J^{\alpha}{}_k J_{jl} - J^{\alpha}{}_l J_{jk} + 2J^{\alpha}{}_j J_{kl}).$$

Let us define the $(1,3)$-tensor $G^i_{jkl} = R^i_{jkl} + 4BK^i_{jkl}$. This new tensor has the same algebraic symmetries as the Riemannian curvature tensor R (including the Bianci identity), in particular, it commutes with the complex structure J:

$$G_{ijkl} = -G_{jikl}\ , \tag{65}$$

$$G_{ijkl} = G_{klij}\ , G^i_{\alpha kl} J^{\alpha}{}_j = J^i{}_{\alpha} G^{\alpha}_{jkl} \tag{66}$$

In addition, from equation (64) it follows, that G^i_{jkl} satisfies

$$a_{i\alpha} G^{\alpha}_{jkl} + a_{\alpha j} G^{\alpha}_{ikl} = 0 \tag{67}$$

for each solution (a_{ij}, λ_i) of equation (3), $a_{ij} \neq \mathrm{const} \cdot g_{ij}$.
Our goal is to show that $G^i_{jkl} \equiv 0$.
For an arbitrary skew-symmetric $(2,0)$-tensor ω^{kl} consider the linear operator

$$G(\omega)^i_j := G^i_{jkl} \omega^{kl}.$$

Since g is hermitian, there exists a basis in T_pM such that the matrices of g and J are given by

$$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varepsilon & \\ & & & \varepsilon \end{pmatrix}, \qquad J = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$$

where $\varepsilon = \pm 1$ depending on the signature of g. Fixing this basis, we will work in matrix notation. Since g is non-trivial, it is important to note that letters J, a and $G(\omega)$ correspond to matrices of linear operators, i.e. $(1,1)$-tensors. By g we denote the matrix of the $(0,2)$-form g_{ij}.

All matrices we are working with commute with the complex structure J. It is a well-known fact (that can be checked by direct calculation) that matrices, commuting with the complex structure, are "complex" in the sence that they have the form

$$\begin{pmatrix} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ -\beta_1 & \alpha_1 & -\beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & \alpha_4 & \beta_4 \\ -\beta_3 & \alpha_3 & -\beta_4 & \alpha_4 \end{pmatrix} \tag{68}$$

Using this form one can define the nondegenerate $\mathbb{R}$-linear mapping ψ

$$\psi : \{Q \in \mathrm{Mat}(4,4,\mathbb{R}) \mid QJ = JQ\} \to \mathrm{Mat}(2,2,\mathbb{C}) \tag{69}$$

given by the formula

$$\psi\left(\begin{pmatrix} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ -\beta_1 & \alpha_1 & -\beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & \alpha_4 & \beta_4 \\ -\beta_3 & \alpha_3 & -\beta_4 & \alpha_4 \end{pmatrix}\right) = \begin{pmatrix} \alpha_1 + i\beta_1 & \alpha_2 + i\beta_2 \\ \alpha_3 + i\beta_3 & \alpha_4 + i\beta_4 \end{pmatrix}.$$

It is easy to check that $\psi(Q_1Q_2) = \psi(Q_1)\psi(Q_2)$ and $\psi(Q^T) = \overline{\psi(Q)}^T$, where "$\overline{}$" denotes the complex conjugation. Moreover, $\psi(J) = i \cdot \mathbf{1}$ and $\psi(g) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$.

To simplify the notation we will identify a matrix with its image under the mapping ψ, for example a and $\psi(a)$ are identified, as well as g and $\psi(g)$.

Since a_{ij} is symmetric, it satisfies the equation

$$ga = \overline{(ga)}^T \tag{70}$$

Thus, there exist real numbers α, β and a complex number Z such that

$$a = \begin{pmatrix} \alpha & Z \\ \bar{Z} & \beta \end{pmatrix}. \tag{71}$$

By assumptions there exist three solutions a_{ij}, A_{ij}, g_{ij} which are linearly independent at the point. Then there exists a nontrivial (i.e., $\neq c \cdot g$ at the point we are working in) solution such that $\alpha = \beta = 0$. Without loss of generality we think that the solution a_{ij} has $\alpha = \beta = 0$ and $Z \neq 0$, i.e.

$$a = \begin{pmatrix} 0 & Z \\ \bar{Z} & 0 \end{pmatrix} \tag{72}$$

Consider now the restrictions that equations (65) and (67) impose on the complex form of $G(\omega)$. Since $G(\omega)_{ij}$ is skew-symmetric

$$gG(\omega) = -\overline{(gG(\omega))}^T.$$

Thus, $G(\omega)$ has the form

$$G(\omega) = \begin{pmatrix} i\alpha & W \\ -\overline{W} & i\beta \end{pmatrix} \tag{73}$$

for certain real numbers α, β and a complex number W. The last condition we have to make use of is

$$aG(\omega) = G(\omega)a$$

Since a is simple (moreover, has different eigenvalues) every matrix that commutes with a is a polynomial of a. (Recall that the matrix a in our convention corresponds to the $(1,1)$-tensor a^i_j.) Thus, $G(\omega) = C \cdot a + D \cdot \mathbf{1}$ for certain complex numbers C and D. Using the explicit form of a and $G(\omega)$ (see (72) and (73)) we obtain

$$\begin{pmatrix} i\alpha & W \\ -\overline{W} & i\beta \end{pmatrix} = C \begin{pmatrix} 0 & Z \\ \bar{Z} & 0 \end{pmatrix} + D \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{74}$$

which implies that both $D = i\alpha = i\beta$ and $C = \frac{W}{Z} = -\overline{\left(\frac{W}{Z}\right)}$ are purely imaginary. Finally we obtain

$$G(\omega) = i \cdot c \cdot a + i \cdot d \cdot \mathbf{1} \tag{75}$$

with real coefficients c, d. If we assume $c \neq 0$, then $G(\omega)$ has different eigenvalues. Thus, $G(\omega)$ is simple. Let us consider another solution A of equation (3). Since it commutes with the simple matrix $G(\omega)$ it is a polynomial of $G(\omega)$:

$$A = \tau G(\omega) + \nu\, \mathbf{1} \tag{76}$$

Subsituting the explicit form of $A = \begin{pmatrix} \alpha_A & Z_A \\ \bar{Z}_A & \beta_A \end{pmatrix}$ we obtain

$$\begin{pmatrix} \alpha_A & Z_A \\ \overline{Z}_A & \beta_A \end{pmatrix} = \tau \begin{pmatrix} i\alpha & W \\ -\overline{W} & i\beta \end{pmatrix} + \nu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{77}$$

which implies that $\tau = i\,t$ is purely imaginary and ν is real. Therefore, equation (76) implies that all solutions of (11) are contained in the 2-dimensional space $itG(\omega) + \nu\,\mathbf{1}$, which gives us the contradiction. Then, $c = 0$. Thus, from (75) we obtain that for every ω the operator $G(\omega)$ is proportional to the complex structure: in the initial "real" notation, we obtain

$$G(\omega)^i_j = d(\omega) J^i{}_j \tag{78}$$

Since the left hand side is linear in ω^{kl}, it follows that $d(\omega) = d_{kl}\omega^{kl}$ and hence $G^i_{jkl}\omega^{kl} = d_{kl}\omega^{kl} J^i{}_j$ implying $G_{ijkl} = d_{kl}J_{ij}$. Using the symmetry relations (65) for G_{ijkl} we obtain $d_{kl}J_{ij} = d_{ij}J_{kl}$ and therefore $d_{kl} = cJ_{kl}$ for some constant $c \neq 0$. Let us show that $G_{ijkl} = cJ_{ij}J_{kl}$ does not satisfy the Bianci identity unless $c = 0$. By direct computation we obtain

$$0 = G_{1234} + G_{1423} + G_{1342} = c(J_{12}J_{34} + J_{14}J_{32} + J_{13}J_{42}) = c(1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0) = c.$$

Thus, $G^i_{jkl} \equiv 0$.

Finally,

$$0 = G^i_{jkl} = R^\alpha_{jkl} + 4BK^\alpha_{jkl},$$

i.e. our metric g has pointwise constant holomorphic curvature $-4B$ (at almost every point, and therefore at every point of M). Thus, M has constant holomorphic sectional curvature (see for example [22, chapter 8]). Then B is a constant and Lemma 6 has been proved for $\dim M = 4$.

□

2.5. **Last step in the proof of Theorem 5.** Above, we proved the following

Statement. *Let (M^{2n}, g, J) be a connected Kähler manifold of dimension $2n \geq 4$. Assume the degree of mobility D of g is ≥ 3. Then, for every solution (a_{ij}, λ_i) of (3) such that $a_{ij} \neq \mathrm{const} \cdot g_{ij}$, almost every point of M has a neighborhood such that in this neighborhood there exists an unique constant B and a scalar function μ such that the "extended" system (6) holds.*

Indeed, the first equation of (6) is equation (3) and is fulfilled everywhere. The second equation is fulfilled almost everywhere by the results of the previous sections. Now, as we noted in Remark 4, at every open set such that the second equation is fulfilled, the third equation is fulfilled as well.

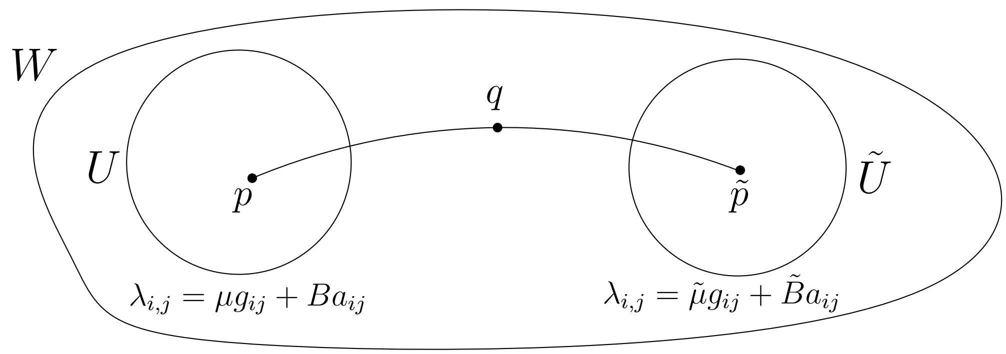

FIGURE 1. There exists q on $\gamma_{p,\tilde{p}}$ such that $\lambda_i = 0$ at q.

Remark 6. The above **statement** is visually close to Theorem 5, the only difference is that in Theorem 5 the constant B and the function μ are universal (i.e., do not depend on the neighborhood). We will prove it in this section.

First let us prove

Lemma 8. *Assume that in every point of an open subset $U \subseteq M$ the extended system (6) holds (for a certain constant B). Then, in this neighborhood, the function $\lambda := \frac{1}{4}a^i_i$ satisfies Tanno's equation*

$$\lambda_{,ijk} = B(2\lambda_{,k}g_{ij} + \lambda_{,i}g_{jk} + \lambda_{,j}g_{ik} - \bar{\lambda}_{,i}J_{jk} - \bar{\lambda}_{,j}J_{ik}). \tag{79}$$

Remark 7. Recall that the differential of the function λ is precisely the covector λ_i from (2), i.e., $\lambda_{,i} = \lambda_i$, see the discussion after Theorem 1.

Proof. If B is a constant, the function μ is smooth as the coefficient of the proportionality of the nonvanishing smooth tensor g_{ij} and the smooth tensor $(\lambda_{i,j} - Ba_{ij})$.

We take the covariant derivative of the second equation of the "extended" system and substitute the first and the third equations inside. In view of $\lambda_i = \lambda_{,i}$, we obtain

$$\begin{aligned}\lambda_{,ijk} &= \mu_{,k} \cdot g_{ij} + Ba_{ij,k} = 2B\lambda_k \cdot g_{ij} + B(\lambda_i g_{jk} + \lambda_j g_{ik} - \bar{\lambda}_i J_{jk} - \bar{\lambda}_j J_{ik}) \\ &= B(2\lambda_k \cdot g_{ij} + \lambda_i g_{jk} + \lambda_j g_{ik} - \bar{\lambda}_i J_{jk} - \bar{\lambda}_j J_{ik}).\end{aligned}$$

□

Now let us prove that the constant B is universal. It is sufficient to prove this in a neighborhood $W(q)$ of an *arbitrary* point q. Indeed, every continuous curve $c : [t_0, t_1] \to M^{2n}$ lies in finite number of such neighborhoods W. Since the constants B for two such intersected neighborhoods must coincide, the value of B at the point $c(t_0)$ equals the value of B at $c(t_1)$. Since the manifold is assumed to be connected, the constant B is therefore universal, i.e., is the same for all neighborhoods.

Let $W \subseteq M$ be a sufficiently small neighborhood. Without loss of generality we can assume that W is geodesically convex, that is, every two points $p, \tilde{p} \in W$ can be connected by a unique geodesic segment lying in W.

We want to show that each two open sets contained in W such that they are as in the **statement** above have the same constant B. Let $U, \tilde{U} \subseteq W$ be nonempty open sets such that in these sets the extended equations (6) are satisfied with constants B for U and $\tilde{B}$ for $\tilde{U}$.

We assume $B \neq \tilde{B}$. We take a point $p \in U$ and connect this point with every point $\tilde{p} \in \tilde{U}$ by a geodesic $\gamma_{p,\tilde{p}} : [0, 1] \to W$, $\gamma_{p,\tilde{p}}(0) = p$, $\gamma_{p,\tilde{p}}(1) = \tilde{p}$ (see Fig. 1).

Let us show that $\gamma_{p,\tilde{p}}$ contains a point q such that $\lambda_i = 0$ at q. Indeed, contracting equation (79) with g^{ij} we obtain

$$\Delta\lambda_{,k} = 4B(n + 1)\lambda_k. \tag{80}$$

If $\lambda_i \neq 0$ at all points of the geodesic $\gamma_{p,\tilde{p}}$, we can find a vector field ξ^i in some neighborhood $U(\gamma_{p,\tilde{p}})$ of the geodesic $\gamma_{p,\tilde{p}}$ such that $\lambda_i \xi^i \neq 0$ at all points of this neighborhood $U(\gamma_{p,\tilde{p}})$. Then, the function

$$\frac{\Delta\lambda_{,k}\xi^k}{4(n+1)\lambda_k\xi^k} \tag{81}$$

is well defined and smooth in $U(\gamma_{p,\tilde{p}})$. Comparing (80) with (81), we see that in a neighborhood of almost every point it is equal to the constant B in this neighborhood, so it is constant on $U(\gamma_{p,\tilde{p}})$. Then, $B = \tilde{B}$ which contradicts our assumption. Finally, there exists a point q of the geodesic $\gamma_{p,\tilde{p}}$ such that $\lambda_i = 0$ at q.

By Corollary 2, $\bar{\lambda}_i$ is a Killing vector field. Then, the function $\dot{\gamma}^i_{p,\tilde{p}}\bar{\lambda}_i$ is constant on the geodesic $\gamma_{p,\tilde{p}}$. Since it vanishes at q, it vanishes at all other points of $\gamma_{p,\tilde{p}}$, in particular we have that at the point $p = \gamma_{p,\tilde{p}}(0)$ the vector $\bar{\lambda}^i$ is orthogonal to $\dot{\gamma}^i_{p,\tilde{p}}(0)$.

The same is true for every geodesic connecting the point p with any other point of $\tilde{U}$. Then, the vector $\bar{\lambda}^i$ at p is orthogonal to many vectors (to all initial vectors of the geodesics starting from p and containing at least one point of $\tilde{U}$); thus $\lambda_i = 0$ at p (see Fig. 2).

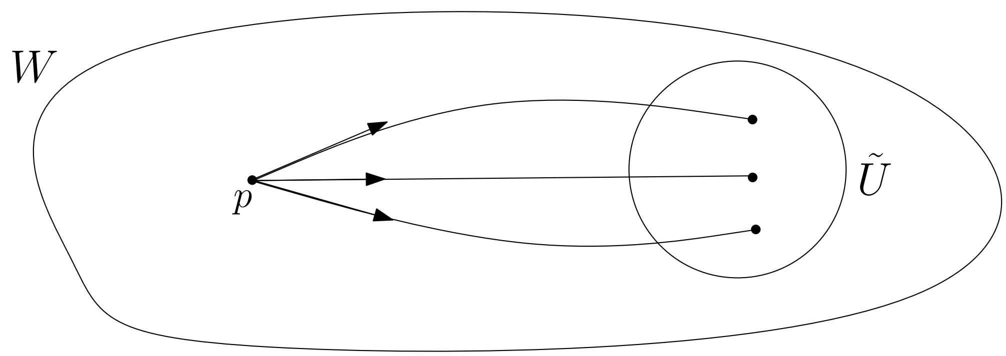

FIGURE 2. $\bar{\lambda}^i$ at p is orthogonal to every $\dot{\gamma}^i_{p,\tilde{p}}$, implying $\lambda^i \equiv 0$.

Replacing the point p by any other point of the neighborhood U, we obtain that $\lambda_i = 0$ at all points of U. By Corollary 3, $\lambda_i \equiv 0$ on the whole manifold. Substituting $\lambda_i \equiv 0$ in the extended system, and using that g_{ij} is not proportional to a_{ij}, we see that $B = 0$ (at almost all points of manifold).

Thus, the constant B is universal on the whole connected manifolds. Theorem 5 is proved.

3. THE CASE $B = 0$

By Corollary 5, we already now that the global constant B, arising in the extended system (6), does not depend on the solutions (a_{ij}, λ_i) of equation (3). In this section we want to investigate the case when $B = 0$. Our goal is to prove the following

Theorem 6. *Let (M^{2n}, g, J) be a closed connected Kähler manifold of dimension $2n \geq 4$ and of degree of mobility ≥ 3. Suppose the constant B in the system (6) is zero, then $\lambda_i \equiv 0$ on the whole M for each solution (a_{ij}, λ_i) of equation (3).*

In particular, every metric $\bar{g}$, h-projectively equivalent to g, is already affinely equivalent to g.

Proof. If $B = 0$, then $\mu = \text{const}$ by the third equation from (6), and the second equations reads $\lambda_{i,j} = \text{const} \cdot g_{ij}$. Then, the hessian $\lambda_{i,j}$ of the function $\lambda := \frac{1}{4}a^i_i$ is covariantly constant.

Since the manifold is closed the function λ has a minimum and a maximum. At a minimum, the Hessian must be non-negatively definite, and at a maximum it must be nonpositively definite. Therefore the Hessian is null, and λ_i is covariantly constant. But as it vanishes at the extremal points, it vanishes everywhere. Thus, $\lambda_i \equiv 0$ as we claim. By Remark 2, every metric $\bar{g}$, h-projectively equivalent to g, is already affine equivalent to g as we claim. $\square$

4. If $B \neq 0$, the metric $-B \cdot g$ is positively definite

Now let us treat the case when the constant B in the system (6) is different from zero. Let (M^{2n}, g, J) be a connected Kähler manifold of dimension $2n \geq 4$. Let (a_{ij}, λ_i, μ) be a solution of (6). Since $B \neq 0$, we can replace g by the metric $-B \cdot g$ (having the same Levi-Civita connection with g).

Then, for every solution (a_{ij}, λ_i, μ) of the system (6), the triple $(-B \cdot a_{ij}, \lambda_i, -\frac{1}{B}\mu)$ is the solution of (6) corresponding to the metric $g' := -B \cdot g$ with the constant $B = -1$. Indeed, the Levi-Civita connections of g and g' coincide, so substituting $(-B \cdot a_{ij}, \lambda_i, -\frac{1}{B}\mu, -Bg, -1)$ instead of $(a_{ij}, \lambda_i, \mu, g, B)$ in the extended system gives the system which is equivalent to the initial extended system.

Note that the mapping $(a_{ij}, \lambda_i, \mu) \mapsto (-B \cdot a_{ij}, \lambda_i, -\frac{1}{B}\mu)$ is linear and bijective, so the degrees of mobility of g and $-Bg$ are equal. Thus, if $B \neq 0$, in the proof of Theorem 2, without loss of generality we can assume that $B = -1$.

The goal of this section is to prove the following

Theorem 7. *Let (M^{2n}, g, J) be a closed connected Kähler manifold of dimension $2n \geq 4$. Suppose (a_{ij}, λ_i, μ) satisfies*

$$\begin{aligned} a_{ij,k} &= \mathcal{J}_{ij}^{i'j'}(\lambda_{i'} g_{j'k} + \lambda_{j'} g_{i'k}) \\ \lambda_{i,j} &= \mu g_{ij} - a_{ij}, \\ \mu_{,i} &= -2\lambda_i \end{aligned} \tag{82}$$

and $\lambda_i \neq 0$ at least at one point. Then, the metric g is positively definite.

Remark 8. The assumption that the manifold is closed is important – one can construct examples of complete pseudo-Riemannian Kähler metrics admitting nontrivial solutions (a_{ij}, λ_i, μ). Simplest examples are pseudo-Riemannian Kähler manifolds of constant holomorphic curvature 4. Examples of nonconstant holomorphic curvature also exist and can be constructed similar to [3, Example 3.1].

We need the following

Lemma 9. *Let (a_{ij}, λ_i, μ) be a solution of the system (82) such that $a_{ij} = 0$, $\lambda_i = 0$, $\mu = 0$ at some point p of the connected Kähler manifold (M^{2n}, g, J).*

Then $a_{ij} \equiv 0$, $\lambda_i \equiv 0$, $\mu \equiv 0$ at all points of M. In particular, the degree of mobility is always finite.

Proof. The system (82) is in the Frobenius form, i.e., the derivatives of the unknowns a_{ij}, λ_i, μ are expressed as (linear) functions of the unknowns:

$$\begin{pmatrix} a_{ij,k} \\ \lambda_{i,j} \\ \mu_{,i} \end{pmatrix} = F \begin{pmatrix} a_{ij} \\ \lambda_i \\ \mu \end{pmatrix},$$

and all linear systems in the Frobenius form have the property that the vanishing of the solution at one point implies the vanishing at all points. □

The rest of this section is dedicated to the proof of Theorem 7. Our first goal will be to show, that it is possible to choose one solution of the system (82) (under the assumptions of Theorem 7) such that the corresponding operator $a^i_j = g^{i\alpha} a_{\alpha j}$ has a clear and simple structure of eigenspaces and eigenvectors.

4.1. Matrix of the extended system.

In order to find the special solution of (82) mentioned above, we rewrite a solution (a_{ij}, λ_i, μ) as a $(1,1)$-tensor on the $(2n+2)$-dimensional manifold $\widehat{M} = \mathbb{R}^2 \times M$ with coordinates $(\underbrace{x_+, x_-}_{\mathbb{R}^2}, \underbrace{x_1, \ldots, x_{2n}}_{M})$. For every solution (a^i_j, λ_i, μ) of the system

(82), let us consider the $(2n+2)\times(2n+2)$-matrix

$$L(a,\lambda,\mu) = \left(\begin{array}{cc|ccc} \mu & 0 & \lambda_1 & \dots & \lambda_{2n} \\ 0 & \mu & \bar\lambda_1 & \dots & \bar\lambda_{2n} \\ \hline \lambda^1 & \bar\lambda^1 & & & \\ \vdots & \vdots & & a^i_j & \\ \lambda^{2n} & \bar\lambda^{2n} & & & \end{array}\right) \tag{83}$$

where $\bar\lambda_i = J^{i'}{}_i\lambda_{i'}$. The matrix $L(a,\lambda,\mu)$ is a well-defined $(1,1)$-tensor field on $\widehat{M}$ (in the sense that after a local coordinate change in M the components of the matrix L transform according to tensor rules).

Remark 9. We consider the metric g_{ij} as a solution of the system (82) with $\lambda_i = 0$ and $\mu = 1$. Thus

$$L(g,0,1) = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & \delta^i_j & \\ 0 & 0 & & & \end{array}\right) = \mathbf{1} \tag{84}$$

Remark 10. We see that the matrix L contains as much information as the triple (a_{ij},λ_i,μ), so in a certain sense it is an alternative equivalent way to write down the triple. In the next section, we will see that the matrix formalism does have advantages: we will show that the polynomials of the matrix L also correspond to certain solutions of the extended system.

Let us also note that there is a visually similar construction in the theory of projectively equivalent metrics, which uses *cone manifolds*, see [32, 33, 3]. However, in the case of h-projectively equivalent metrics, the extended operator is not covariantly constant (as in the theory of projectively equivalent metrics) which poses additional difficulties.

4.2. **Algebraic properties of L.** A linear combination of two matrices of the form (83) is also a matrix of this form, and corresponds to the linear combination of the solutions (with the same coefficients). The next lemma shows that the k-th power of the matrix also corresponds to a solution of the extended system.

Lemma 10. *Let (a,λ,μ) be a solution of (82). Then, for every $k\geq 0$ there exists a solution $(\tilde a,\tilde\lambda,\tilde\mu)$ such that*

$$L^k(a,\lambda,\mu) = L(\tilde a,\tilde\lambda,\tilde\mu), \text{ where } L^k = \underbrace{L\cdot\ldots\cdot L}_{k\ times}.$$

Proof. Given two solutions (a,λ,μ) and $(A,\Lambda,\mathcal{M})$ of (82), let us calculate the product of the corresponding matrices $L(a,\lambda,\mu)$ and $L(A,\Lambda,\mathcal{M})$: by direct calculations we obtain

$$L(a,\lambda,\mu)\cdot L(A,\Lambda,\mathcal{M}) = \tag{85}$$
$$= \left(\begin{array}{cc|ccc} \mu\mathcal{M}+\lambda_k\Lambda^k & \lambda_k\bar\Lambda^k & \mu\Lambda_1+\lambda_kA^k_1 & \dots & \mu\Lambda_{2n}+\lambda_kA^k_{2n} \\ \bar\lambda_k\Lambda^k & \mu\mathcal{M}+\lambda_k\Lambda^k & \mu\bar\Lambda_1+\bar\lambda_kA^k_1 & \dots & \mu\bar\Lambda_{2n}+\bar\lambda_kA^k_{2n} \\ \hline \mathcal{M}\lambda^1+a^1_k\Lambda^k & \mathcal{M}\bar\lambda^1+a^1_k\bar\Lambda^k & & & \\ \vdots & \vdots & & a^i_kA^k_j+\lambda^i\Lambda_j+\bar\lambda^i\bar\Lambda_j & \\ \mathcal{M}\lambda^{2n}+a^{2n}_k\Lambda^k & \mathcal{M}\bar\lambda^{2n}+a^{2n}_k\bar\Lambda^k & & & \end{array}\right)$$

Suppose that

$$\mu\Lambda_j+\lambda_kA^k_j = \mathcal{M}\lambda_j+a^k_j\Lambda_k \quad\text{and}\quad \lambda^k\bar\Lambda_k = 0 \tag{86}$$

then

$$L(a,\lambda,\mu)\cdot L(A,\Lambda,\mathcal{M}) = L(\underbrace{a^i_kA^k_j+\lambda^i\Lambda_j+\bar\lambda^i\bar\Lambda_j}_{\tilde a_{ij}}, \underbrace{\mu\Lambda_i+\lambda_kA^k_i}_{\tilde\lambda_i}, \underbrace{\mu\mathcal{M}+\lambda_k\Lambda^k}_{\tilde\mu}) \tag{87}$$

Now we show that the operator $L(\tilde a, \tilde\lambda, \tilde\mu)$ is self-adjoint and $\tilde a$, $\tilde\lambda$ and $\tilde\mu$ satisfy (82).

Indeed, let us check the first equation of (82):

(88)
$$\begin{aligned}\tilde a_{ij,k} = (a_{is}A^s_j + \lambda_i\Lambda_j + \bar\lambda_i\bar\Lambda_j)_{,k} = a_{is,k}A^s_j + a^s_iA_{sj,k} + \lambda_{i,k}\Lambda_j + \lambda_i\Lambda_{j,k} + \bar\lambda_{i,k}\bar\Lambda_j + \bar\lambda_i\bar\Lambda_{j,k} \overset{(82)}{=}\\ = A^s_j\lambda_ig_{sk} + A^s_j\lambda_sg_{ik} + A^s_j\bar\lambda_iJ^{s'}{}_sg_{s'k} + A^s_j\bar\lambda_sJ^{i'}{}_ig_{i'k}+\\ + a^s_i\Lambda_jg_{sk} + a^s_i\Lambda_sg_{jk} + a^s_i\bar\Lambda_jJ^{s'}{}_sg_{s'k} + a^s_i\bar\lambda_sJ^{j'}{}_jg_{j'k}+\\ + \mu g_{ik}\Lambda_j - a_{ik}\Lambda_j + \mathcal{M}g_{jk}\lambda_i - A_{jk}\lambda_i + \mu J^{i'}{}_ig_{i'k}\bar\Lambda_j - J^{i'}{}_ia_{i'k}\bar\Lambda_j + \mathcal{M}J^{j'}{}_jg_{j'k}\bar\lambda_i - J^{j'}{}_jA_{j'k}\bar\lambda_i =\\ = g_{ik}(\lambda_sA^s_j + \mu\Lambda_j) + g_{jk}(\Lambda_sa^s_i + \mathcal{M}\lambda_i) + J^{i'}{}_ig_{i'k}(\bar\lambda_sA^s_j + \mu\bar\Lambda_j) + J^{j'}{}_jg_{j'k}(\bar\Lambda_sa^s_i + \mathcal{M}\bar\lambda_i) \overset{(86)}{=}\\ = \mathcal{J}^{i'j'}_{ij}(\tilde\lambda_{i'}g_{j'k} + \tilde\lambda_{j'}g_{i'k})\end{aligned}$$

For the second equation one can calculate:

(89)
$$\begin{aligned}\tilde\lambda_{i,k} = (\mu\Lambda_i + \lambda_jA^j_i)_{,k} = \mu_{,k}\Lambda_i + \mu\Lambda_{i,k} + \lambda_{j,k}A^j_i + \lambda^jA_{ij,k} \overset{(82)}{=}\\ = -2\lambda_k\Lambda_i+\mu\mathcal{M}g_{ik}-\mu A_{ik}+\mu A_{ik}-a_{jk}A^j_i+\lambda^j\Lambda_ig_{jk}+\lambda^j\Lambda_jg_{ik}+\lambda^jJ^{j'}{}_jJ^{i'}{}_i\Lambda_{i'}g_{j'k}+\lambda^jJ^{j'}{}_jJ^{i'}{}_i\Lambda_{j'}g_{i'k} =\\ = (\mu\mathcal{M} + \lambda^j\Lambda_j)g_{ik} - (\lambda_k\Lambda_i + \bar\lambda_k\bar\Lambda_i + A_{ij}a^j_k) + \lambda^j\bar\Lambda_jJ^{i'}{}_ig_{i'k} \overset{(86)}{=} \tilde\mu g_{ki} - \tilde a_{ki}\end{aligned}$$

From this equation we see that $\tilde a_{ij}$ is symmetric as a linear combination of two symmetric tensors. The last equation of (82) reads

(90)
$$\begin{aligned}\tilde\mu_{,i} = (\mu\mathcal{M} + \lambda_k\Lambda^k)_{,i} = \mu_{,i}\mathcal{M} + \mu\mathcal{M}_{,i} + \lambda_{k,i}\Lambda^k + \lambda^k\Lambda_{k,i} =\\ \overset{(82)}{=} -2\lambda_i\mathcal{M}-2\Lambda_i\mu+\Lambda^k(\mu g_{ik}-a_{ik})+\lambda^k(\mathcal{M}g_{ik}-A_{ik}) = -(\mu\Lambda_i+\lambda_kA^k_i)-(\mathcal{M}\lambda_i+\Lambda_ka^k_i) \overset{(86)}{=} -2\tilde\lambda_i\end{aligned}$$

Thus, $(\tilde a, \tilde\lambda, \tilde\mu)$ is a solution of (82).

Let us now show that the operator $L(A, \Lambda, \mathcal{M}) = L^k(a, \lambda, \mu)$ satisfies the conditions (86).

Since $L^k \cdot L = L \cdot L^k$, using (85) we obtain

$$\mu\Lambda_j + \lambda_kA^k_j = \mathcal{M}\lambda_j + a^k_j\Lambda_k$$

The last condition will be checked by induction. Suppose $\lambda^i\bar\Lambda_i = 0$ then

$$\lambda^iJ^{i'}{}_i\widetilde\Lambda_{i'} = \lambda^i \cdot J^{i'}{}_i(\mu\Lambda_{i'} + \lambda_kA^k_{i'}) = \mu\cdot 0 + \lambda^k(J^{i'}{}_iA_{ki'})\lambda^i = 0$$

which completes the proof of Lemma 10. □

From Lemma 10, we immediately obtain

Corollary 6. *Let (a_{ij}, λ_i, μ) be a solution of (82) and $P(t) = c_kt^k + \cdots + c_0$ be an arbitrary polynomial with real coefficients. Then there exists a solution $(A_{ij}, \Lambda_i, \mathcal{M})$ of (82) such that*

$$L(A_{ij}, \Lambda_i, \mathcal{M}) = c_k \cdot L^k(a_{ij}, \lambda_i, \mu) + \cdots + \mathbf{1} := P(L(a_{ij}, \lambda_i, \mu)),$$

where $\mathbf{1}$ is the identity $(2n+2)\times(2n+2)-$matrix.

4.3. **There exists a solution $(\check a_{ij}, \check\lambda_i, \check\mu)$ such that $L(\check a_{ij}, \check\lambda_i, \check\mu)$ is a projector.** We assume that $(M^{2n\geq 4}, g, J)$ is a closed connected Kähler manifold. Our goal is to show that the existence of a solution (a_{ij}, λ_i, μ) of (82) such that $\lambda_i \not\equiv 0$ implies the existence of a solution $(\check a_{ij}, \check\lambda_i, \check\mu)$ of (82) such that the matrix $L(\check a_{ij}, \check\lambda_i, \check\mu)$ is a non-trivial (i.e. $\neq 0$ and $\neq \mathbf{1}$) projector. (Recall that a matrix L is a *projector*, if $L^2 = L$.). We need

Lemma 11. *Let (M^{2n}, g, J) be a connected Kähler manifold and (a_{ij}, λ_i, μ) be a solution of (82).*

Let $P(t)$ be the minimal polynomial of $L(a, \lambda, \mu)$ at the point $\hat p \in \widehat M$. Then, $P(t)$ is the minimal polynomial of $L(a, \lambda, \mu)$ at every $\hat q \in \widehat M$.

Convention. *We will always assume that the leading coefficient of a minimal polynomial is* 1.

Proof. As we have already proved, there exists a solution $(\tilde{a}_{ij}, \tilde{\lambda}_i, \tilde{\mu})$ such that

$$P(L(a, \lambda, \mu)) = L(\tilde{a}, \tilde{\lambda}, \tilde{\mu}).$$

Since $P(L(a, \lambda, \mu))$ vanishes at the point $\hat{p} = (x_+, x_-, p)$, then $\tilde{a} = 0$, $\tilde{\lambda} = 0$ and $\tilde{\mu} = 0$ at p. Then, by Lemma 9, the solution $(\tilde{a}_{ij}, \tilde{\lambda}_i, \tilde{\mu})$ is identically zero on M. Thus, $P(L(a, \lambda, \mu))$ vanishes at all points of $\widehat{M}$. It follows, that the polynomial $P(t)$ is divisible by the minimal polynomial $Q(t)$ of $L(a, \lambda, \mu)$ at $\hat{q}$. By the same reasoning (interchanging $\hat{p}$ and $\hat{q}$), we obtain that $Q(t)$ is divisible by $P(t)$. Consequently, $P(t) = Q(t)$. □

Corollary 7. *The eigenvalues of $L(a, \lambda, \mu)$ are constant functions on $\widehat{M}$.*

Proof. By Lemma 11, the minimal polynomial does not depend on the point of $\widehat{M}$. Then, the roots of the minimal polynomial are also constant (i.e., do not depend on the point of $\widehat{M}$). □

In order to find the desired special solution of the system (82), we will use that M is closed.

Lemma 12. *Suppose (M^{2n}, g, J) is a closed connected Kähler manifold. Let (a_{ij}, λ_i, μ) be a solution of (82) such that $\lambda_i \neq 0$ at least at one point. Then, at every point of $\widehat{M}$ the matrix $L(a, \lambda, \mu)$ has at least two different real eigenvalues.*

Proof. Since M is closed, the function μ admits its maximal and minimal values $\mu_{\max}$ and $\mu_{\min}$. Let $p \in M$ be a point where $\mu = \mu_{\max}$. At this point, $\mu_{,i} = 0$ implying $\lambda_i = \bar{\lambda}_i = 0$ in view of the third equation of (82). Then, the matrix of $L(a, \lambda, \mu)$ at p has the form

$$L(a, \lambda, \mu) = \left(\begin{array}{cc|ccc} \mu_{\max} & 0 & 0 & \dots & 0 \\ 0 & \mu_{\max} & 0 & \dots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & a^i_j & \\ 0 & 0 & & & \end{array}\right) \tag{91}$$

Thus, $\mu_{\max}$ is an eigenvalue of $L(a, \lambda, \mu)$ at p and, since the eigenvalues are constant, $\mu_{\max}$ is an eigenvalue of $L(a, \lambda, \mu)$ at every point of M. The same holds for $\mu_{\min}$. Since $\lambda_i \not\equiv 0$, μ is not constant implying $\mu_{\max} \neq \mu_{\min}$. Finally, $L(a, \lambda, \mu)$ has two different real eigenvalues $\mu_{\max}, \mu_{\min}$ at every point. □

Remark 11. For further use let us note that in the proof of Lemma 12 we have proved that if $\mu_{,i} = 0$ at a point p then $\mu(p)$ is an eigenvalue of L.

Finally, let us show that there is always a solution of (82) of the desired special kind:

Lemma 13. *Suppose (M^{2n}, g, J) is a closed and connected Kähler manifold. For every solution (a_{ij}, λ_i, μ) of (82) such that λ_i is not identically zero on M, there exists a polynomial $P(t)$ such that $P(L(a, \lambda, \mu))$ is a non-trivial (i.e. it is neither* **0** *nor* **1***) projector.*

Proof. We take a point $\hat{p} \in \widehat{M}$. By Lemma 12, $L(a_{ij}, \lambda_i, \mu)$ has at least two real eigenvalues at the point $\hat{p}$. Then, by linear algebra, there exists a polynomial P such that $P(L(a_{ij}, \lambda_i, \mu))$ is a nontrivial projector at the point p. Evidently, a matrix C is a nontrivial projector, if and only if its minimal polynomial is $t(t-1)$ (multiplied by any nonzero constant). Since by Lemma 11 the minimal polynomial of $P(L(a_{ij}, \lambda_i, \mu))$ is the same at all points, the matrix $P(L(a_{ij}, \lambda_i, \mu))$ is a projector at every point of M. □

Thus, (under the assumptions of Theorem 7), without loss of generality we can think that a solution of the system (82) on a closed and connected Kähler manifold M with degree of mobility ≥ 3 is chosen such that the corresponding L is a projector.

4.4. **Structure of eigenspaces of a^i_j, if $L(a,\lambda,\mu)$ is a nontrivial projector.** We assume that $L(a,\lambda,\mu)$ is a nontrivial projector. Then, it has precisely two eigenvalues: 1 and 0 and the $(2n+2)$-dimensional tangent space of $\widehat{M}$ at every point $\hat{x} = (x_+, x_-, p)$ can be decomposed into the sum of the corresponding eigenspaces

$$T_{\hat{x}}\widehat{M} = E_{L(a,\lambda,\mu)}(1) \oplus E_{L(a,\lambda,\mu)}(0).$$

The dimensions of $E_{L(a,\lambda,\mu)}(1)$ and of $E_{L(a,\lambda,\mu)}(0)$ are even; we assume that the dimension of $E_{L(a,\lambda,\mu)}(1)$ is $2k+2$ and the dimension of $E_{L(a,\lambda,\mu)}(0)$ is $2n-2k$.

By Lemma 12, $\mu_{\max}$ and $\mu_{\min}$ are eigenvalues of $L(a,\lambda,\mu)$. Then, $\mu_{\min} = 0 \le \mu(x) \le 1 = \mu_{\max}$ on M. In view of Remark 11, the only critical values of μ are 1 and 0.

Lemma 14. *Let (a_{ij}, λ_i, μ) be a solution of (82) such that $L(a,\lambda,\mu)$ is a non-trivial projector. Then, the following statements hold:*

1. *At a point p such that $0 < \mu < 1$, a^i_j has the following structure of eigenvalues and eigenspaces*
   (a) *eigenvalue 1 with geometric multiplicity $2k$;*
   (b) *eigenvalue 0 with geometric multiplicity $(2n-2k-2)$;*
   (c) *eigenvalue $(1-\mu)$ with multiplicity 2.*
2. *At a point p such that $\mu = 1$, a^i_j has the following structure of eigenvalues and eigenspaces:*
   (a) *eigenvalue 1 with geometric multiplicity $2k$;*
   (b) *eigenvalue 0 with geometric multiplicity $(2n-2k)$;*
3. *At a point p such that $\mu = 0$, a^i_j has the following structure of eigenvalues and eigenspaces:*
   (a) *eigenvalue 1 with geometric multiplicity $2k+2$;*
   (b) *eigenvalue 0 with geometric multiplicity $(2n-2k-2)$.*

Convention. *We identify M with the set $(0,0)\times M \subset \widehat{M}$. This identification allows us to consider T_xM as a linear subspace of $T_{(0,0)\times x}\widehat{M}$: the vector $(v_1, ..., v_n) \in T_xM$ is identified with $(0,0,v_1,...,v_n) \in T_{(0,0)\times x}\widehat{M}$.*

Proof. For any vector $v \in E_1 = E_{L(a,\lambda,\mu)}(1) \cap TM$ we calculate

$$L(a,\lambda,\mu)v = \left(\begin{array}{cc|ccc} \mu & 0 & \lambda_1 & \dots & \lambda_{2n} \\ 0 & \mu & \bar\lambda_1 & \dots & \bar\lambda_{2n} \\ \hline \lambda^1 & \bar\lambda^1 & & & \\ \vdots & \vdots & & a^i_j & \\ \lambda^{2n} & \bar\lambda^{2n} & & & \end{array}\right) \begin{pmatrix} 0 \\ 0 \\ v^1 \\ \vdots \\ v^{2n} \end{pmatrix} = \begin{pmatrix} \lambda_j v^j \\ \bar\lambda_j v^j \\ \\ a^i_j v^j \\ \\ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v^1 \\ \vdots \\ v^{2n} \end{pmatrix} \tag{92}$$

Thus, $v = (v^1, \dots v^{2n})$ is an eigenvector of a^i_j with eigenvalue 1. Moreover, it is orthogonal to both λ^i and $\bar\lambda^i$. Similarly, any $v \in E_0 = E_{L(a,\lambda,\mu)}(0) \cap T_xM$ is an eigenvector of a^i_j with eigenvalue 0 and it is orthogonal to λ^i and $\bar\lambda^i$. Note that the dimension of E_1 is at least $\dim E_{L(a,\lambda,\mu)}(1) - 2 = 2k$, and the dimension of E_0 is at least $\dim E_{L(a,\lambda,\mu)}(0) - 2 = 2n-2k-2$.

Thus, at every point x there are three pairwise orthogonal subspaces in T_xM: E_1, E_0 and $span\{\lambda^i, \bar\lambda^i\}$.

If $0 < \mu < 1$ at x, $\lambda_i \ne 0$ by Remark 11. Then, the dimension of $E_1 \oplus E_0 \oplus \mathrm{span}\{\lambda^i, \bar\lambda^i\}$ is at least $2n-2k-2+2k+2 = 2n$. Since $E_1 \oplus E_0 \oplus \mathrm{span}\{\lambda^i, \bar\lambda^i\} \subseteq T_xM$, the dimension of E_1 is $2n-2k-2$ and the dimension of E_0 is $2k$, and $E_1 \oplus E_0 \oplus \mathrm{span}\{\lambda^i, \bar\lambda^i\} = T_xM$.

Let us now show that λ^i and $\bar\lambda^i$ are eigenvectors of a^i_j with the eigenvalue $(1-\mu)$. We multiply the first basis vector $(1,0,\dots,0)$ by the matrix $L(a,\lambda,\mu)^2 - L(a,\lambda,\mu)$ (which is identically zero). We obtain

$$0 = (L(a,\lambda,\mu)^2 - L(a,\lambda,\mu)) \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \mu^2 + \lambda_j\lambda^j - \mu \\ \bar\lambda_i\lambda^i \\ \\ \mu\lambda^i + a^i_j\lambda^j - \lambda^i \\ \\ \end{pmatrix} \tag{93}$$

This gives us the necessary equation $a^i_j\lambda^j = (1-\mu)\lambda^i$.

Finally, we have that T_xM is the direct sum $E_1 \oplus E_0 \oplus \mathrm{span}\{\lambda^i, \bar\lambda^i\}$; E_1 consists of eigenvectors of a^i_j with eigenvalue 1 and has dimension $2n-2k-2$, E_1 consists of eigenvectors of a^i_j with eigenvalue 0 and has dimension $2k$; $\mathrm{span}\{\lambda^i, \bar\lambda^i\}$ consists of eigenvectors of a^i_j with eigenvalue $(1-\mu)$ and has dimension 2, as we claimed in the first statement of the lemma.

The proof at the points x such that $\mu(x) = 0$ or $\mu(x) = 1$ is similar (and is easier), and will be left to the reader. □

4.5. **If there exists a solution (a, λ, μ) of the system (82) corresponding to a non-trivial projector, the metric g is positively definite on M (assumed closed).** Above we have proved that, under the assumptions of Theorem 7, there always exists a solution (a_{ij}, λ_i, μ) of (82) such that the corresponding matrix $L(a, \lambda, \mu)$ is a non-trivial projector, implying that the eigenvalues and the dimension of eigenspaces of a^i_j is given by Lemma 14. Now we are ready to prove that g is positively definite (as we claimed in Theorem 7).

Let us consider such a solution (a_{ij}, λ_i, μ). We rewrite the second equation in (82) in the form

$$\mu_{,ij} = 2a_{ij} - 2\mu\, g_{ij} \tag{94}$$

Let p be a point where μ takes its maximal value 1. As we have already shown, $\lambda^i(p) = 0$ and the tangent space T_pM is the direct sum of the eigenspaces of a^i_j:

$$T_pM = E_1 \oplus E_0$$

Consider the restriction of (94) to E_0. Since the restriction of the bilinear form a_{ij} to E_0 is identically zero, the restriction of (94) to E_0 reads

$$\mu_{,ij}|_{E_0} = -2\ g_{ij}|_{E_0}.$$

Now, $\mu_{,ij}$ is the Hessian of μ at the maximum point p. Then, it is non-positively definite. Hence, the non-degenerate metric tensor g_{ij} is positively definite on E_0 at p. Let us now consider the distribution of the orthogonal complement $E_1^\perp$, which is well-defined, smooth and integrable on $\{x \in M \mid \mu(x) > 0\}$. The restriction of the metric g to $E_1^\perp$ is non-degenerate at the points of $\{x \in M \mid \mu(x) > 0\}$. Since at the point p $E_1^\perp$ coincides with E_0, it is positively definite at p. Hence, by continuity, it is positively definite at the connected component of $\{x \in M \mid \mu(x) > 0\}$ containing p. Since every connected component of $\{x \in M \mid \mu(x) > 0\}$ has a point such that $\mu = 1$, the restriction of the metric g to $E_1^\perp$ is positively definite at all points of $\{x \in M \mid \mu(x) > 0\}$.

Similarly, at a minimum point q one can consider the restriction of (94) to E_1:

$$\mu_{,ij}|_{E_1} = 2\ a_{ij}|_{E_1} = 2\ g_{ij}|_{E_1},$$

since $a^i_{j\,|E_1} = \delta^i_{j\,|E_1}$. Then, g is positively definite on E_1 at q. Considering the distribution $E_0^\perp$, we obtain that the restriction of g to $E_0^\perp$ is positively definite at $\{x \in M \mid \mu(x) < 1\}$.

Evidently, the sets $\{x \in M \mid \mu(x) < 1\}$ and $\{x \in M \mid \mu(x) > 0\}$ have an nonempty intersection. At every point x of the intersection, $T_xM = E_0^\perp + E_1^\perp$. Since the restriction of the metric to $E_0^\perp$ and to $E_1^\perp$ is positively definite, the metric is positively definite as we claimed. Theorem 7 is proved.

5. Tanno-Theorem completes the proof of Theorem 1

We assume that (M^{2n}, g, J) is a closed connected Kähler manifold of dimension $2n \geq 4$ with degree of mobility $D \geq 3$. Let $\bar g$ be a metric h-projectively equivalent to g. We consider the corresponding solution (a_{ij}, λ_i, μ) of the extended system. If the metric $\bar g$ is not affinely equivalent to g, by Theorem 6 we obtain $B \neq 0$. As we explained in the beginning of Section 4, by multiplication of the metric by a nonzero constant, we can achieve $B = -1$. Without loss of generality, we think that $B = -1$. By Theorem 7, the metric g is Riemannian.

Now, by Lemma 8, the function $\lambda := \frac{1}{4}a^i_i$ satisfies the equation

$$\lambda_{,ijk} + (2\lambda_{,k}g_{ij} + \lambda_{,i}g_{jk} + \lambda_{,j}g_{ik} + (J^\alpha{}_i J^\beta{}_j + J^\alpha{}_j J^\beta{}_i)\lambda_{,\alpha}g_{\beta k}) = 0, \tag{95}$$

moreover, by Remark 2, if $\bar g$ is not affinely equivalent to g, the function λ is not a constant.

As we recalled in Section 1.8, this equation was considered in [53]. Tanno has proved, that the existence of a non-constant solution of this equation on a closed connected Riemannian manifold implies that the metric g has positive constant holomorphic sectional curvature equal to 4 (see [53, Theorem 10.5], and also Section 1.8). Then, (M^{2n}, g, J) is $(\mathbb{C}P(n), 4 \cdot g_{FS}, J_{standard})$ as we claimed. Theorem 2 is proved.

Remark 12. As we already mentioned in Section 1.9, in Sections 3, 4 we did not actually use the assumption that the degree of mobility is ≥ 3: we used the system (6) only. Thus, the following statement holds:

Let $(M^{2n\geq 4}, g, J)$ be a closed connected Kähler manifold. Assume there exists a solution (a_{ij}, λ_i, μ) of (6) such that $\lambda_i \not\equiv 0$. Then, (M^{2n}, g, J) is $(\mathbb{C}P(n), \mathrm{const} \cdot g_{FS}, J_{standard})$ (for a certain const $\neq 0$*).*

6. Proof of Theorem 4: equation (5) is equivalent to system (6)

In Lemma 8, we have shown that for a solution of the extended system (6) equation (5) is fulfilled. We will now show that a nonconstant solution of (5) allows us to construct a solution (a_{ij}, λ_i, μ) of the extended system (6) (with $B = \kappa$ and $\lambda_i = f_{,i} \not\equiv 0$) provided that the manifold is closed.

Let f be a non-constant solution of equation (5) on a closed connected manifold M. Then, $\kappa \neq 0$. Indeed, we can proceed as in Section 3: if $\kappa = 0$, then equation (5) reads $f_{,ijk} = 0$. Then, the hessian $f_{,ij}$ of the function f is covariantly constant. Since the manifold is closed, the function f has a minimum and a maximum. At a minimum, the Hessian must be non-negatively definite, and at a maximum it must be nonpositively definite. Therefore the Hessian is null, and $f_{,i}$ is covariantly constant. But as it vanishes at the extremal points, it vanishes everywhere. Thus, $f = \mathrm{const}$ contradicting the assumptions.

Consider the symmetric, hermitian tensor a_{ij} defined by the following formula:

$$a_{ij} = \frac{1}{\kappa} f_{,ij} - 2 f g_{ij} \tag{96}$$

Let us check that $(a_{ij}, \lambda_i = f_{,i}, \mu = 2\kappa f)$ satisfies (6) with $B = \kappa$. Indeed, covariantly differentiating $a_{ij} = \frac{1}{\kappa} f_{,ij} - 2 f g_{ij}$ and substituting (5), we obtain

$$\begin{aligned} a_{ij,k} = \frac{1}{\kappa} f_{,ijk} - 2 f_{,k} g_{ij} = 2 f_{,k} g_{ij} + f_{,i} g_{jk} + f_{,j} g_{ik} - \bar{f}_{,i} J_{jk} - \bar{f}_{,j} J_{ik} - 2 f_{,k} g_{ij} = \\ = f_{,i} g_{jk} + f_{,j} g_{ik} - \bar{f}_{,i} J_{jk} - \bar{f}_{,j} J_{ik}, \end{aligned} \tag{97}$$

which is the first equation of (6). The second equation of (6) is equivalent to (96), the third equation is fulfilled by the construction. Since f is non-constant, $\lambda_i = f_{,i} \not\equiv 0$. Now, as we proved in Section 4, the metric $-\mathrm{sgn}(B) \cdot g$ is positively definite. Finally, for positively definite metrics, Theorem 4 was proved by Tanno in [53]. Theorem 4 is proved.

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